

DE BRANGES-ROVNYAK SPACES AND NORM-CONSTRAINT INTERPOLATION

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ABSTRACT. For S a contractive analytic operator-valued function on the unit disk \mathbb{D} , de Branges and Rovnyak associate a Hilbert space of analytic functions $\mathcal{H}(S)$. A companion survey provides equivalent definitions and basic properties of these spaces as well as applications to function theory and operator theory. The present survey brings to the fore more recent applications to a variety of more elaborate function theory problems, including H^∞ -norm constrained interpolation, connections with the Potapov method of Fundamental Matrix Inequalities, parametrization for the set of all solutions of an interpolation problem, variants of the Abstract Interpolation Problem of Katsnelson, Kheifets, and Yuditskii, boundary behavior and boundary interpolation in de Branges-Rovnyak spaces themselves, and extensions to multivariable and Kreĭn-space settings.

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1. INTRODUCTION

In the late 1960s and early 1970s, Louis de Branges and James Rovnyak introduced and studied spaces of vector-valued holomorphic functions on the open unit disk \mathbb{D} associated with what is now called a Schur-class function $S \in \mathcal{S}(\mathcal{U}, \mathcal{V})$ (i.e., a holomorphic function S on the unit disk with values equal to contraction operators between Hilbert coefficient spaces \mathcal{U} and \mathcal{V}). Motivation for the study of these spaces came from quantum scattering theory (see [dBR1966a, dB1977, dBS1968]), and operator model theory for Hilbert space contraction operators and the invariant subspace problem (see [dBR1966a, Appendix] and [dBR1966b]).

Interpolation by Schur-class functions is an older area which appeared first within geometric function theory. Over the years there have been a variety of approaches to the study of Schur-class functions and associated interpolation problems (e.g., Schur algorithm, iterated one-step extension procedures, transfer-function realization techniques, the Grassmannian Kreĭn-space geometry approach, reproducing kernel Hilbert space methods, and commutant-lifting methods to mention a few). The general topic for this survey article is de Branges-Rovnyak spaces; hence the focus here is only on those approaches which rely to some extent on de Branges-Rovnyak spaces.

There are now at least three distinct ways of introducing the de Branges-Rovnyak spaces:

- (1) the original definition of de Branges and Rovnyak (as the complementary space of $S \cdot H^2$),
- (2) as the range of the Toeplitz defect operator with lifted norm,
or
- (3) as the reproducing kernel Hilbert space with reproducing kernel given by the de Branges-Rovnyak positive kernel.

2. DE BRANGES-ROVNYAK SPACES

In what follows, the symbol $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ stands for the space of bounded linear operators mapping a Hilbert space \mathcal{U} into a Hilbert space \mathcal{Y} , abbreviated to $\mathcal{L}(\mathcal{Y})$ in case $\mathcal{U} = \mathcal{Y}$. The notation $H^2(\mathcal{Y})$ is used to denote the standard Hardy space of \mathcal{Y} -valued functions on the open unit disk \mathbb{D} with square-summable sequence of Taylor coefficients while $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ denotes the Schur class of functions analytic on \mathbb{D} with values equal to contractive operators in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$. The de Branges-Rovnyak space $\mathcal{H}(S)$ associated with a given Schur-class function $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ was originally defined as the complementary space of $S \cdot H^2$ by the prescription

$$\mathcal{H}(S) = \{f \in H^2(\mathcal{U}) : \|f\|_{\mathcal{H}(S)}^2 := \sup_{g \in H^2(\mathcal{U})} \{\|f + Sg\|_{H^2(\mathcal{Y})}^2 - \|g\|_{H^2(\mathcal{U})}^2\} < \infty\}. \quad (2.1)$$

In particular, it follows from (2.1) that $\|f\|_{\mathcal{H}(K_S)} \geq \|f\|_{H^2(\mathcal{Y})}$ for every $f \in \mathcal{H}(K_S)$, i.e., that $\mathcal{H}(K_S)$ is contained in $H^2(\mathcal{Y})$ contractively.

Two equivalent definitions of de Branges-Rovnyak spaces (more convenient in certain contexts) involve the notion of a *reproducing kernel Hilbert space* which will be now recalled.

2.1. Reproducing kernel Hilbert spaces. A *reproducing kernel Hilbert space* (RKHS) is a Hilbert space whose elements are functions on some set Ω with values in a coefficient Hilbert space, say \mathcal{Y} , such that the evaluation map $\mathbf{e}(\omega) : f \mapsto f(\omega)$ is continuous from \mathcal{H} into \mathcal{Y} for each $\omega \in \Omega$. Associated with any such space is a positive $\mathcal{L}(\mathcal{Y})$ -valued kernel on Ω , i.e., a function $K : \Omega \times \Omega \rightarrow \mathcal{L}(\mathcal{Y})$ with the positive-kernel property

$$\sum_{i,j=1}^N \langle K(\omega_i, \omega_j) y_j, y_i \rangle_{\mathcal{Y}} \geq 0 \quad (2.2)$$

for any choice of finitely many points $\omega_1, \dots, \omega_N \in \Omega$ and vectors $y_1, \dots, y_N \in \mathcal{Y}$, which “reproduces” the values of the functions in \mathcal{H} in the sense that

- (i) the function $\omega \mapsto K(\omega, \zeta)y$ is in \mathcal{H} for each $\zeta \in \Omega$ and $y \in \mathcal{Y}$,
and
- (ii) the reproducing formula

$$\langle f, K(\cdot, \zeta)y \rangle_{\mathcal{H}} = \langle f(\zeta), y \rangle_{\mathcal{Y}}$$

holds for all $f \in \mathcal{H}$, $\zeta \in \Omega$, and $y \in \mathcal{Y}$.

An early thorough treatment of RKHSs (for the case $\mathcal{Y} = \mathbb{C}$) is the paper of Aronszajn [A1950]; a good recent treatment is in the book

[AMcC2002], while the recent paper [BV2003] formulates more general settings (formal commuting or noncommuting variables).

Given a pair of reproducing kernel Hilbert spaces $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ where say $\mathcal{H}(K_1)$ consists of functions with values in \mathcal{U} and $\mathcal{H}(K_2)$ consists of functions with values in \mathcal{Y} , an object of much interest for operator theorists is the space of multipliers $\mathcal{M}(K_1, K_2)$ consisting of $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ -valued functions F on Ω with the property that the multiplication operator

$$M_F: f(\zeta) \mapsto F(\zeta)f(\zeta)$$

maps $\mathcal{H}(K_1)$ into $\mathcal{H}(K_2)$. The simple computation

$$\begin{aligned} \langle M_F f, K(\cdot, \zeta)y \rangle_{\mathcal{H}(K_2)} &= \langle F(\zeta)f(\zeta), y \rangle_{\mathcal{Y}} \\ &= \langle f(\zeta), F(\zeta)^*y \rangle_{\mathcal{U}} = \langle f, K_1(\cdot, \zeta)F(\zeta)^*y \rangle_{\mathcal{H}(K_1)} \end{aligned}$$

shows that

$$(M_F)^*: K_2(\cdot, \zeta)y \mapsto K_1(\cdot, \zeta)F(\zeta)^*y. \quad (2.3)$$

Therefore

$$\langle (I - M_F M_F^*)K_2(\cdot, \zeta)y, K_2(\cdot, \omega)y' \rangle_{\mathcal{H}(K_2)} = \langle (K_2(\omega, \zeta) - F(\zeta)K_1(\omega, \zeta)F(\zeta)^*)y, y' \rangle_{\mathcal{Y}}$$

which implies that F is a contractive multiplier from $\mathcal{H}(K_1)$ to $\mathcal{H}(K_2)$ if and only if the kernel $K_2(\omega, \zeta) - F(\zeta)K_1(\omega, \zeta)F(\zeta)^*$ is positive on $\Omega \times \Omega$. Letting $K_1(\omega, \zeta) \equiv I_{\mathcal{Y}}$ and performing a rescaling leads to the following proposition [BeBu1984].

Proposition 2.1. *A function $F : \Omega \rightarrow \mathcal{Y}$ belongs to $\mathcal{H}(K)$ with $\|F\|_{\mathcal{H}(K)} \leq \gamma$ if and only if the kernel $K(\omega, \zeta) - \gamma^{-2}F(\zeta)F(\zeta)^*$ is positive on $\Omega \times \Omega$.*

2.2. The Toeplitz operator characterization of $\mathcal{H}(K)$. A first example of a reproducing kernel Hilbert space is the Hardy space $H^2(\mathcal{Y})$ of \mathcal{Y} -valued functions on the open unit disk \mathbb{D} with square-summable sequence of Taylor coefficients. This space can be viewed as a RKHS with the Szegő kernel tensored with the identity operator on \mathcal{Y} : $k_{\text{Sz}}(z, \zeta)I_{\mathcal{Y}}$ where $k_{\text{Sz}}(z, \zeta) = \frac{1}{1-z\bar{\zeta}}$. The space of multipliers $\mathcal{M}(k_{\text{Sz}}I_{\mathcal{U}}, k_{\text{Sz}}I_{\mathcal{Y}})$ between two Hardy spaces can be identified with the space $H^\infty(\mathcal{U}, \mathcal{Y})$ of bounded analytic functions on \mathbb{D} with values in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ while the set of contractive multipliers is identified with the Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$. Indeed, for $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ and for any $f \in H^2(\mathcal{U})$,

$$\begin{aligned} \|Sf\|_{H^2(\mathcal{Y})}^2 &= \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} \|S(re^{it})f(re^{it})\|_{\mathcal{Y}}^2 dt \\ &\leq \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} \|S(e^{it})f(e^{it})\|_{\mathcal{Y}}^2 dt = \|f\|_{H^2(\mathcal{U})}^2 \end{aligned} \quad (2.4)$$

which shows that M_S is a contraction from $H^2(\mathcal{U})$ to $H^2(\mathcal{Y})$. The general complementation theory applied to the contractive operator M_S provides the *characterization of $\mathcal{H}(K_S)$ as the operator range*

$$\mathcal{H}(K_S) = \text{Ran}(I - M_S M_S^*)^{\frac{1}{2}} \quad (2.5)$$

with the lifted norm

$$\|(I - M_S M_S^*)^{\frac{1}{2}} f\|_{\mathcal{H}(K_S)} = \|(I - \pi)f\|_{H^2(\mathcal{Y})} \quad (2.6)$$

for all $f \in H^2(\mathcal{Y})$ where π is the orthogonal projection onto $\text{Ker}(I - M_S M_S^*)^{\frac{1}{2}}$. Upon setting $f = (I - M_S M_S^*)^{\frac{1}{2}} h$ in (2.6) one gets

$$\|(I - M_S M_S^*)h\|_{\mathcal{H}(K_S)} = \langle (I - M_S M_S^*)h, h \rangle_{H^2(\mathcal{Y})}. \quad (2.7)$$

2.3. Reproducing kernel characterization of $\mathcal{H}(S)$. As a result of the general identity (2.3),

$$M_S^*: k_{S_z}(\cdot, \zeta)y \mapsto k_{S_z}(\cdot, \zeta)S(\zeta)^*y \quad (2.8)$$

and hence

$$(I - M_S M_S^*)k_{S_z}(z, \zeta)y = K_S(z, \zeta)y \quad (2.9)$$

where

$$K_S(z, \zeta) = (I - S(z)S(\zeta)^*)k_{S_z}(z, \zeta) = \frac{I_{\mathcal{Y}} - S(z)S(\zeta)^*}{1 - z\bar{\zeta}} \quad (2.10)$$

is the *de Branges-Rovnyak kernel* associated to the given $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$. Application of inequality (2.4) to $f = \sum_{j=1}^N k_{S_z}(\cdot, w_j)y_j \in H^2(\mathcal{Y})$ leads one, on account of (2.9), to

$$\sum_{i,j=1}^N \langle K_S(\omega_i, \omega_j)y_j, y_i \rangle_{\mathcal{Y}} \geq 0,$$

and it follows that K_S is a positive kernel on $\mathbb{D} \times \mathbb{D}$. Combining the characterization (2.5) and equality (2.9) one can see that $K_S(\cdot, \zeta)y \in \mathcal{H}(S)$ for each $\zeta \in \mathbb{D}$ and $y \in \mathcal{Y}$, and also, for $f = (I - M_S M_S^*)f_1 \in \mathcal{H}(S)$,

$$\begin{aligned} \langle f, K_S(\cdot, \zeta)y \rangle_{\mathcal{H}(S)} &= \langle f, (I - M_S M_S^*)(k_{S_z}(\cdot, \zeta)y) \rangle_{\mathcal{H}(S)} \\ &= \langle f, k_{S_z}(\cdot, \zeta)y \rangle_{H^2(\mathcal{Y})} = \langle f(\zeta), y \rangle_{\mathcal{Y}} \end{aligned}$$

from which it follows that $\mathcal{H}(S)$ is a *reproducing kernel Hilbert space with reproducing kernel equal to $K_S(z, \zeta)$* (2.10). This characterization of $\mathcal{H}(S)$ turns out to be quite convenient in interpolation and realization contexts. This section concludes by recording several useful facts concerning de Branges-Rovnyak spaces collected in the following theorem.

Theorem 2.2. *If $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, the space $\mathcal{H}(S)$ has the following properties:*

- (1) $\mathcal{H}(S)$ is a linear space, indeed a reproducing kernel Hilbert space with reproducing kernel $K_S(z, w)$ given by

$$K_S(z, w) = \frac{I - S(z)S(w)^*}{1 - z\bar{w}}.$$

- (2) The space $\mathcal{H}(S)$ is invariant under the backward-shift operator

$$R_0: f(z) \mapsto [f(z) - f(0)]/z \quad (2.11)$$

and the following norm estimate holds:

$$\|R_0 f\|_{\mathcal{H}(S)}^2 \leq \|f\|_{\mathcal{H}(S)}^2 - \|f(0)\|_{\mathcal{Y}}^2. \quad (2.12)$$

Moreover, equality holds in (2.12) for all $f \in \mathcal{H}(S)$ if and only if $\mathcal{H}(S)$ has the property

$$S(z) \cdot u \in \mathcal{H}(S) \Rightarrow S(z) \cdot u \equiv 0.$$

- (3) For any $u \in \mathcal{U}$, the function $R_0(Su)$ is in $\mathcal{H}(S)$. If one lets $\tau: \mathcal{U} \rightarrow \mathcal{H}(S)$ denote the operator

$$\tau: u \mapsto R_0(Su) = \frac{S(z) - S(0)}{z} u, \quad (2.13)$$

then the adjoint R_0^* of the operator R_0 (2.11) on $\mathcal{H}(S)$ is given by

$$R_0^*: f(z) \mapsto zf(z) - S(z) \cdot \tau^*(f) \quad (2.14)$$

with the following formula for the norm holding:

$$\|R_0^* f\|_{\mathcal{H}(S)}^2 = \|f\|_{\mathcal{H}(S)}^2 - \|\tau^*(f)\|_{\mathcal{U}}^2. \quad (2.15)$$

- (4) Let \mathbf{U}_S be the colligation matrix given by

$$\mathbf{U}_S = \begin{bmatrix} A_S & B_S \\ C_S & D_S \end{bmatrix} := \begin{bmatrix} R_0 & \tau \\ \mathbf{e}(0) & S(0) \end{bmatrix} : \begin{bmatrix} \mathcal{H}(S) \\ \mathcal{U} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}(S) \\ \mathcal{Y} \end{bmatrix} \quad (2.16)$$

where R_0 and τ are given by (2.11) and (2.13) and where $\mathbf{e}(0): \mathcal{H}(S) \rightarrow \mathcal{Y}$ is the evaluation-at-zero map:

$$\mathbf{e}(0): f(z) \mapsto f(0).$$

Then \mathbf{U}_S is coisometric, and one recovers $S(z)$ as the characteristic function of \mathbf{U}_S :

$$S(z) = D_S + zC_S(I - zA_S)^{-1}B_S. \quad (2.17)$$

3. DE BRANGES-ROVNYAK SPACES AND SCHUR-CLASS INTERPOLATION

This section will show how de Branges-Rovnyak spaces appear in a natural way in the context of Schur-class interpolation theory. The main idea comes from the work of Katsneslon, Kheifets and Yuditskii [Ka1985, Ka1997, KKY1987, Kh1998, KY1994], and is closely connected with Potapov's method of Fundamental Matrix Inequalities [Ka1997, KY1994, Ko1974, Ko1975, KoP1974, KoP1982].

The starting point is a relatively simple left-tangential operator-valued version of the classical Nevanlinna-Pick problem which consists of the following: *Given n distinct points $z_1, \dots, z_n \in \mathbb{D}$ and given vectors $E_1, \dots, E_n \in \mathcal{Y}$ and $N_1, \dots, N_n \in \mathcal{U}$, find a Schur-class function $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ (if such exists) such that*

$$S(z_i)^* E_i = N_i \quad \text{for } i = 1, \dots, n. \quad (3.1)$$

In what follows, E_i^* and N_i^* will be viewed as elements of $\mathcal{L}(\mathcal{Y}, \mathbb{C})$ and $\mathcal{L}(\mathcal{U}, \mathbb{C})$, respectively. Upon multiplying both parts in (3.1) by $k_{Sz}(\cdot, z_i)$ and making use of formula (2.8) one concludes that (3.1) can be written equivalently in terms of the Toeplitz operator T_S as

$$T_S^* : E_i(1 - z\bar{z}_i)^{-1} \mapsto N_i(1 - z\bar{z}_i)^{-1} \quad \text{for } i = 1, \dots, n$$

or equivalently, as the single condition

$$T_S^* : \sum_{i=1}^n E_i(1 - z\bar{z}_i)^{-1} x_i \mapsto \sum_{i=1}^n N_i(1 - z\bar{z}_i)^{-1} x_i \quad (3.2)$$

holding for all $x_1, \dots, x_n \in \mathbb{C}$. Introduce the operators

$$T = \begin{bmatrix} \bar{z}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{z}_n \end{bmatrix}, \quad E = [E_1 \quad \dots \quad E_n], \quad N = [N_1 \quad \dots \quad N_n] \quad (3.3)$$

and two observability operators $\mathcal{O}_{E,T} : \mathbb{C}^n \rightarrow H^2(\mathcal{Y})$ and $\mathcal{O}_{N,T} : \mathbb{C}^n \rightarrow H^2(\mathcal{U})$ defined as

$$\mathcal{O}_{E,T} : x \mapsto E(I - zT)^{-1}x \quad \text{and} \quad \mathcal{O}_{N,T} : x \mapsto N(I - zT)^{-1}x. \quad (3.4)$$

It is readily seen from (3.3), (3.4) that condition (3.2) can be equivalently written in the operator form as

$$T_S^* \mathcal{O}_{E,T} = \mathcal{O}_{N,T}. \quad (3.5)$$

In general the pair of operators (E, T) (where say $E \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $T \in \mathcal{L}(\mathcal{X})$) is said to be *output stable* if the associated observability

operator as in (3.4) maps \mathcal{X} into $H^2(\mathcal{Y})$. This discussion suggests the more general interpolation problem:

IP: *Given Hilbert space operators $T \in \mathcal{L}(\mathcal{X})$, $E \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $N \in \mathcal{L}(\mathcal{X}, \mathcal{U})$ such that the pairs (E, T) and (N, T) are output-stable, find a Schur-class function $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ subject to interpolation condition (3.5).*

The Nevanlinna-Pick problem recalled above is a particular case of the problem **IP** corresponding to $\mathcal{X} = \mathbb{C}^n$ and to the special choice (3.3) of the operators T , E and N .

Observe that for an output-stable pair (E, T) and for any $x \in \mathcal{X}$,

$$\|\mathcal{O}_{E,T}x\|_{H^2(\mathcal{Y})}^2 - \|\mathcal{O}_{E,T}Tx\|_{H^2(\mathcal{Y})}^2 = \sum_{k=0}^{\infty} \|ET^kx\|_{\mathcal{Y}}^2 - \sum_{k=0}^{\infty} \|ET^{k+1}x\|_{\mathcal{Y}}^2 = \|Ex\|_{\mathcal{Y}}^2$$

and similarly,

$$\|\mathcal{O}_{N,T}x\|_{H^2(\mathcal{U})}^2 - \|\mathcal{O}_{N,T}Tx\|_{H^2(\mathcal{U})}^2 = \|Nx\|_{\mathcal{U}}^2.$$

Define

$$P := \mathcal{O}_{E,T}^* \mathcal{O}_{E,T} - \mathcal{O}_{N,T}^* \mathcal{O}_{N,T}. \quad (3.6)$$

Then it follows that

$$\begin{aligned} \langle Px, x \rangle_{\mathcal{X}} - \langle PTx, Tx \rangle_{\mathcal{X}} &= \|\mathcal{O}_{E,T}x\|_{H^2(\mathcal{Y})}^2 - \|\mathcal{O}_{N,T}x\|_{H^2(\mathcal{U})}^2 \\ &\quad - \|\mathcal{O}_{E,T}Tx\|_{H^2(\mathcal{Y})}^2 + \|\mathcal{O}_{N,T}Tx\|_{H^2(\mathcal{U})}^2 \\ &= \|Ex\|_{\mathcal{Y}}^2 - \|Nx\|_{\mathcal{U}}^2 \quad \text{for all } x \in \mathcal{X} \end{aligned}$$

which can be written in operator form as

$$P - TPT^* = EE^* - NN^*. \quad (3.7)$$

The operator P defined above from interpolation data is called *the Pick operator* of the problem **IP**. Observe, that in case (3.3) of the tangential Nevanlinna-Pick problem, P admits the explicit matrix formula

$$P = \left[\frac{\langle E_i, E_j \rangle_{\mathcal{Y}} - \langle N_i, N_j \rangle_{\mathcal{U}}}{1 - z_i \bar{z}_j} \right]_{i,j=1}^n. \quad (3.8)$$

If the problem **IP** has a solution (say, $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$), then equality (3.5) holds for a contraction operator T_S^* and therefore,

$$\|\mathcal{O}_{N,T}x\|_{H^2(\mathcal{U})}^2 = \|T_S^* \mathcal{O}_{E,T}x\|_{H^2(\mathcal{U})}^2 \leq \|\mathcal{O}_{E,T}x\|_{H^2(\mathcal{Y})}^2 \quad \text{for all } x \in \mathcal{X}$$

which simply means that the Pick operator (3.6) is positive semidefinite. The necessity part of the next result follows from this discussion.

Theorem 3.1. *The problem **IP** has a solution if and only if its Pick matrix is positive semidefinite:*

$$P := \mathcal{O}_{E,T}^* \mathcal{O}_{E,T} - \mathcal{O}_{N,T}^* \mathcal{O}_{N,T} \geq 0. \quad (3.9)$$

Remark 3.2. Taking adjoints in (3.5) gives $\mathcal{O}_{E,T}^* T_S = \mathcal{O}_{N,T}^*$ where operators on both sides map $H^2(\mathcal{U})$ into $H^2(\mathcal{Y})$. Upon restricting this operator equality to the coefficient space \mathcal{U} (that is, to the space of constant functions in $H^2(\mathcal{U})$) one gets

$$\mathcal{O}_{E,T}^* M_S|_{\mathcal{U}} = \mathcal{O}_{N,T}^*|_{\mathcal{U}} = N^*. \quad (3.10)$$

The latter condition is a consequence of (3.5). However, it can be equivalently used in the formulation of the **IP** for the following reason: *if the pair (E, T) is output stable and equality (3.10) holds for a Schur-class function $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, then the pair (N, T) is also output stable (so that the observability operator $\mathcal{O}_{N,T}$ maps \mathcal{X} into $H^2(\mathcal{U})$) and equality (3.5) holds.*

At this point de Branges-Rovnyak spaces come into play. With any Schur-class function $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, one can associate the linear map $F^S : \mathcal{X} \rightarrow H^2(\mathcal{Y})$ by the formula

$$F^S : x \mapsto (\mathcal{O}_{E,T} - T_S \mathcal{O}_{N,T}) x. \quad (3.11)$$

If S satisfies condition (3.5), then

$$F^S x = (\mathcal{O}_{E,T} - T_S T_S^* \mathcal{O}_{E,T}) x = (I - T_S T_S^*) \mathcal{O}_{E,T} x$$

and therefore $F^S x$ belongs to $\mathcal{H}(S)$ by characterization (2.5). Moreover,

$$\begin{aligned} \|F^S x\|_{\mathcal{H}(S)}^2 &= \langle (I - T_S T_S^*) \mathcal{O}_{E,T}, \mathcal{O}_{E,T} \rangle_{H^2(\mathcal{Y})} \\ &= \langle (\mathcal{O}_{E,T}^* \mathcal{O}_{E,T} - \mathcal{O}_{N,T}^* \mathcal{O}_{N,T}) x, x \rangle_{\mathcal{X}} = \langle P x, x \rangle_{\mathcal{X}} = \|P^{\frac{1}{2}} x\|_{\mathcal{X}}^2. \end{aligned}$$

It has been shown that under the assumption (3.11), a function $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ is a solution to the problem **IP** only if the linear transformation (3.11) maps \mathcal{X} into $\mathcal{H}(S)$ with equality $\|F^S x\|_{\mathcal{H}(S)}^2 = \|P^{\frac{1}{2}} x\|_{\mathcal{X}}^2$ for every $x \in \mathcal{X}$. The converse ("if") statement was established in [KKY1987]. This and several other characterizations of solutions to the problem **IP** are presented in the next theorem. In some statements, the function S will not be assumed to be in the Schur class; consequently the notation $M_S : f \mapsto S f$ rather than T_S will be used for the operator of multiplication by S .

Theorem 3.3. *Assume that condition (3.9) is satisfied and let F^S be defined as in (3.11) (with M_S instead of T_S) for a function $S : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{Y})$. The following are equivalent:*

- (1) S is a solution of the problem **IP**.
- (2) $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ and the function $F^S x$ belongs to $\mathcal{H}(S)$ and satisfies

$$\|F^S x\|_{\mathcal{H}(S)} = \|P^{\frac{1}{2}} x\|_{\mathcal{X}} \quad \text{for every } x \in \mathcal{X}. \quad (3.12)$$

- (3) $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ and the function $F^S x$ belongs $\mathcal{H}(S)$ and satisfies

$$\|F^S x\|_{\mathcal{H}(S)} \leq \|P^{\frac{1}{2}} x\|_{\mathcal{X}} \quad \text{for every } x \in \mathcal{X}. \quad (3.13)$$

- (4) The following kernel is positive in $\mathbb{D} \times \mathbb{D}$:

$$\mathbb{K}_S(z, \zeta) = \begin{bmatrix} P & (I_{\mathcal{X}} - \bar{\zeta} T^*)^{-1} (E^* - N^* S(\zeta)^*) \\ (E - S(z)N)(I_{\mathcal{X}} - zT)^{-1} & \frac{I_{\mathcal{Y}} - S(z)S(\zeta)^*}{1 - z\bar{\zeta}} \end{bmatrix} \succeq 0. \quad (3.14)$$

- (5) F^S maps \mathcal{X} into $H^2(\mathcal{Y})$ and the operator

$$\mathbf{P} := \begin{bmatrix} P & (F^S)^* \\ F^S & I - M_S M_S^* \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ H^2(\mathcal{Y}) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X} \\ H^2(\mathcal{Y}) \end{bmatrix} \quad (3.15)$$

is positive semidefinite.

Proof. A brief sketch will be given. Implication (1) \Rightarrow (2) was demonstrated above. Implication (2) \Rightarrow (3) is trivial. Implication (3) \Rightarrow (4) follows from Proposition 2.1. Implication (4) \Rightarrow (5) follow from the identity

$$\langle \mathbf{P}f, f \rangle_{\mathcal{X} \oplus H^2(\mathcal{Y})} = \sum_{j, \ell=1}^r \left\langle \mathbb{K}_S(z_j, z_\ell) \begin{bmatrix} x_\ell \\ y_\ell \end{bmatrix}, \begin{bmatrix} x_j \\ y_j \end{bmatrix} \right\rangle_{\mathcal{X} \oplus \mathcal{Y}}$$

holding for every vector $f \in \mathcal{X} \oplus H^2(\mathcal{Y})$ of the form

$$f = \sum_{j=1}^r \begin{bmatrix} x_j \\ k_{S, z_j}(\cdot, z_j) y_j \end{bmatrix} \quad (x_j \in \mathcal{X}, y_j \in \mathcal{Y}, z_j \in \mathbb{D}).$$

For implication (5) \Rightarrow (1), first observe that since $I - M_S M_S^*$ is positive semidefinite (equivalently, M_S is a contraction) then $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ and $M_S = T_S$. By definitions (3.6) and (3.11),

$$\mathbf{P} = \begin{bmatrix} \mathcal{O}_{E,T}^* \mathcal{O}_{E,T} - \mathcal{O}_{N,T}^* \mathcal{O}_{N,T} & \mathcal{O}_{E,T}^* - \mathcal{O}_{N,T}^* T_S^* \\ \mathcal{O}_{E,T} - T_S \mathcal{O}_{N,T} & I - T_S T_S^* \end{bmatrix} \geq 0.$$

By the standard Schur complement argument, the latter inequality is equivalent to

$$\hat{\mathbf{P}} := \begin{bmatrix} I_{H^2(\mathcal{U})} & \mathcal{O}_{N,T} & T_S^* \\ \mathcal{O}_{N,T}^* & \mathcal{O}_{E,T}^* \mathcal{O}_{E,T} & \mathcal{O}_{E,T}^* \\ T_S & \mathcal{O}_{E,T} & I_{H^2(\mathcal{Y})} \end{bmatrix} \geq 0,$$

since \mathbf{P} is the Schur complement of the block $I_{H^2(\mathcal{U})}$ in $\hat{\mathbf{P}}$. On the other hand, the latter inequality holds if and only if the Schur complement of the block $I_{H^2(\mathcal{Y})}$ in $\hat{\mathbf{P}}$ is positive semidefinite:

$$\begin{bmatrix} I_{H^2(\mathcal{U})} & \mathcal{O}_{N,T} \\ \mathcal{O}_{N,T}^* & \mathcal{O}_{E,T}^* \mathcal{O}_{E,T} \end{bmatrix} - \begin{bmatrix} T_S^* \\ \mathcal{O}_{E,T}^* \end{bmatrix} [T_S \quad \mathcal{O}_{E,T}] \geq 0.$$

One can write the latter inequality as

$$\begin{bmatrix} I_{H^2(\mathcal{U})} - M_S^* M_S & \mathcal{O}_{N,T} - T_S^* \mathcal{O}_{E,T} \\ \mathcal{O}_{N,T}^* - \mathcal{O}_{E,T}^* T_S & 0 \end{bmatrix} \geq 0$$

and arrive at $\mathcal{O}_{E,T}^* T_S = \mathcal{O}_{N,T}$ which means that S is a solution of **IP**. \square

It can be shown that Theorem 3.3 holds in a more general setting of contractive multipliers from one reproducing kernel Hilbert space into another [Bo2003].

3.1. V.P. Potapov's method of Fundamental Matrix Inequalities. Theorem 3.3 originates in the approach suggested by V. P. Potapov in early 1970s and developed later by his collaborators and followers. The method consisted of three parts: given an interpolation problem,

- (1) establish the solvability criterion in terms of the Pick operator P of the problem and establish the identity (the "fundamental identity" in Potapov's terminology) satisfied by this P ;
- (2) characterize all solutions S to the problem in terms of the "fundamental matrix inequality" $\mathbf{K} = \begin{bmatrix} P & \star \\ \star & \star \end{bmatrix} \geq 0$ where \mathbf{K} is certain structured matrix depending on the unknown function S and having P as a diagonal block;
- (3) describe all solutions S of the inequality $\mathbf{K} \geq 0$ using factorization methods.

One of the main reasons to develop this method was that in the completely indeterminate case (where P is strictly positive definite), the operator-valued problem can be settled in much the same way as in the scalar-valued case. The method was tested on a number of classical interpolation problems [Du1982, Ka1985, Ko1974, Ko1975, KoP1974, KoP1982] and then was largely unified and extended in [KKY1987] (see also [KY1994]). Problem **IP** can be used to illustrate Potapov's

method as follows. The solvability criterion is given in (3.9) in terms of P which satisfies the "fundamental identity" (3.7). The next step is presented in the theorem below.

Theorem 3.4. *Let P be defined as in (3.9). A function $S : \mathbb{D} \mapsto \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is a solution to the problem **IP** if and only if it is analytic on \mathbb{D} and the following matrix is positive semidefinite for all $z \in \mathbb{D}$:*

$$\begin{bmatrix} P & (I_{\mathcal{X}} - \bar{z}T^*)^{-1}(E^* - N^*S(z)^*) \\ (E - S(z)N)(I_{\mathcal{X}} - zT)^{-1} & \frac{I_{\mathcal{Y}} - S(z)S(z)^*}{1 - |z|^2} \end{bmatrix} \geq 0. \quad (3.16)$$

The proof (for the case where \mathcal{X} , \mathcal{Y} and \mathcal{U} are all finite dimensional) can be found in [BD1998, Section 3]. The "if" part is a fairly straightforward consequence of the Schwarz-Pick inequality (of course, this part follows also from Theorem 3.3, since the matrix in (3.16) is nothing else but $\mathbb{K}_S(z, z)$ and therefore condition (3.14) is stronger than (3.16)). The "only if" part is much trickier. Interpolation conditions are derived from (3.16) using a special transformation of the latter inequality suggested first in [KKY1987] (see also [Ka1997] for a related survey). Further developments showed that it is much more convenient to work with positive kernels rather than positive semidefinite matrices. Besides, as one can see from Theorem 3.3, the "kernel" setting makes connections between Nevanlinna-Pick type interpolation problems and de Branges-Rovnyak spaces more transparent.

3.2. The analytic Abstract Interpolation Problem. The very formulation of the problem **IP** requires that the observability operators $\mathcal{O}_{E,T}$ and $\mathcal{O}_{N,T}$ be bounded from \mathcal{X} into $H^2(\mathcal{Y})$ and $H^2(\mathcal{U})$ respectively. Besides, the special form (3.6) of the operator P is essential for proving implication (5) \Rightarrow (1) in Theorem 3.3. However, upon close inspection, one can see that the equivalences (3) \Leftrightarrow (4) \Leftrightarrow (5) in Theorem 3.3 survive under weaker assumptions that P is any positive semidefinite operator on \mathcal{X} and that

- (a) *The function $\begin{bmatrix} E \\ N \end{bmatrix} (I - zT)^{-1}x$ is holomorphic on \mathbb{D} for each $x \in \mathcal{X}$.*

For reasons explained below, it should also be required that

- (b) *P is a positive semidefinite solution to the Stein equation (3.7),*

and formulate the Abstract Interpolation Problem as follows:

AIP: Given the data $\{E, N, T, P\}$ subject to assumptions (a), (b), find all $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ such that for every $x \in \mathcal{X}$, the function

$$(F^S x)(z) = (E - S(z)N)(I_{\mathcal{X}} - zT)^{-1}x \quad (3.17)$$

belongs to the de Branges-Rovnyak space $\mathcal{H}(S)$ and satisfies the norm constraint $\|F^S x\|_{\mathcal{H}(S)} \leq \|P^{\frac{1}{2}}x\|_{\mathcal{X}}$.

The latter problem is a left-tangential adaptation of the more general bi-tangential Abstract Interpolation Problem formulated in [KKY1987] (see also [Kh1998] for an overview) in terms of a more elaborate two-component version of the de Branges-Rovnyak space $\tilde{\mathcal{D}}(S)$ (a good reference for the formulation of this two-component space is [NV1989] as well as the survey article companion to this one [BB2014]). The present survey does not treat this more general interpolation problem.

The next result can be arrived at via a careful inspection of the proof of Theorem 3.3.

Theorem 3.5. *Let P, T, E and N satisfy assumptions (a), (b). Then a function $S : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is a solution of the **AIP** if and only if the kernel $\mathbb{K}_S(z, \zeta)$ of the form (3.14) is positive on $\mathbb{D} \times \mathbb{D}$.*

An important example of a concrete interpolation problem which is a particular case of the problem **AIP** but not of the **IP** is the boundary interpolation problem [BK2008a].

3.3. Parametrization of the solution set. The third step of the Potapov method is to describe all functions S such that the matrix (3.16) is positive semidefinite or, equivalently, such that the kernel (3.14) is positive on $\mathbb{D} \times \mathbb{D}$. This was first done for the case where the Pick operator P is strictly positive definite (in early developments, all the problems were matrix-valued and with finitely many interpolation conditions, so P was a matrix which was assumed be positive definite). If P is strictly positive definite, then it follows from factorization

$$\mathbb{K}_S(z, \zeta) = \begin{bmatrix} I & 0 \\ F^S(z)P^{-1} & I \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & K_S(z, \zeta) - F^S(z)P^{-1}F^S(\zeta)^* \end{bmatrix} \begin{bmatrix} I & P^{-1}F^S(\zeta)^* \\ 0 & I \end{bmatrix}$$

that (3.14) holds if and only if the kernel

$$\tilde{K}_S(z, \zeta) = K_S(z, \zeta) - F^S(z)P^{-1}F^S(\zeta)^*$$

is positive on $\mathbb{D} \times \mathbb{D}$. Using the definitions (2.10) and (3.17) of K_S and F^S and making use of the operators

$$J = \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & -I_{\mathcal{U}} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} E \\ N \end{bmatrix}, \quad (3.18)$$

one can represent the kernel \tilde{K}_S as

$$\begin{aligned} \tilde{K}_S(z, \zeta) &= \frac{I_{\mathcal{Y}} - S(z)S(\zeta)^*}{1 - z\bar{\zeta}} \\ &\quad - (E - S(z)N)(I - zT)^{-1}P^{-1}(I - \bar{\zeta}T^*)^{-1}(E^* - N^*S(\zeta)^*) \\ &= [I \quad -S(z)] \left\{ \frac{J}{1 - z\bar{\zeta}} - C(I - zT)^{-1}P^{-1}(I - \bar{\zeta}T^*)^{-1}C^* \right\} \begin{bmatrix} I \\ -S(\zeta)^* \end{bmatrix}. \end{aligned} \quad (3.19)$$

The crucial step is to find a function $\Theta : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{Y} \oplus \mathcal{U})$ such that

$$\frac{J - \Theta(z)J\Theta(\zeta)^*}{1 - z\bar{\zeta}} = C(I - zT)^{-1}P^{-1}(I - \bar{\zeta}T^*)^{-1}C^* \quad (z, \zeta \in \mathbb{D}). \quad (3.20)$$

If $\text{spec}T \cap \mathbb{T} \neq \mathbb{T}$, i.e., if there exists a boundary point $\mu \in \mathbb{T}$ such that $(\mu I - T^*)^{-1} \in \mathcal{L}(\mathcal{X})$, one may try to find a Θ normalized by $\Theta(\mu) = I_{\mathcal{Y} \oplus \mathcal{U}}$. Letting $\zeta = \mu$ in (3.20) gives

$$J - \Theta(z)J = (1 - z\bar{\mu})C(I - zT)^{-1}P^{-1}(I - \bar{\mu}T^*)^{-1}C^*$$

which then implies

$$\begin{aligned} \Theta(z) &= I - (1 - z\bar{\mu})C(I - zT)^{-1}P^{-1}(I - \bar{\mu}T^*)^{-1}C^*J \\ &= I + (z - \mu)C(I - zT)^{-1}P^{-1}(\mu I - T^*)^{-1}C^*J, \end{aligned}$$

and eventually, on account of (3.18),

$$\begin{aligned} \Theta(z) &= \begin{bmatrix} \Theta_{11}(z) & \Theta_{12}(z) \\ \Theta_{21}(z) & \Theta_{22}(z) \end{bmatrix} \\ &= I + (z - \mu) \begin{bmatrix} E \\ N \end{bmatrix} (I - zT)^{-1}P^{-1}(\mu I - T^*)^{-1} \begin{bmatrix} E^* & -N^* \end{bmatrix}. \end{aligned} \quad (3.21)$$

The accomplishment so far is a function satisfying (3.20) for every $z \in \mathbb{D}$ and a fixed $\zeta = \mu \in \mathbb{T}$. A straightforward calculation based solely on the Stein identity (3.7) shows that the function (3.21) actually satisfies the identity (3.20) for all $z, \zeta \in \mathbb{D}$. Moreover, another calculation (again based on the identity (3.7) only) shows that

$$\frac{J - \Theta(z)^*J\Theta(\zeta)}{1 - \bar{z}\zeta} = \tilde{C}(I - \bar{z}T^*)^{-1}P(I - \zeta T)^{-1}\tilde{C}^* \quad (3.22)$$

where

$$\tilde{C} = JC(I - \mu T)^{-1}P^{-1}(\mu I - T^*).$$

Formulas (3.20) and (3.21) show that the function Θ is J -bicontractive, i.e., that

$$\Theta(z)J\Theta(z)^* \leq J \quad \text{and} \quad \Theta(z)^*J\Theta(z) \leq J \quad \text{for all } z \in \mathbb{D}. \quad (3.23)$$

Another method of constructing a J -contractive Θ subject to the identity (3.20) is based on the Kreĭn space arguments.

Lemma 3.6. *Let P be a strictly positive solution to the Stein equation*

$$P - T^*PT = E^*E - N^*N = C^*JC. \quad (3.24)$$

Then there exists an injective operator $\begin{bmatrix} B \\ D \end{bmatrix} : \mathcal{Y} \oplus \mathcal{U} \rightarrow \mathcal{X} \oplus \mathcal{Y} \oplus \mathcal{U}$ such that

$$\begin{bmatrix} T & B \\ C & D \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} T^* & C^* \\ B^* & D^* \end{bmatrix} = \begin{bmatrix} P^{-1} & 0 \\ 0 & J \end{bmatrix}, \quad (3.25)$$

$$\begin{bmatrix} T^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} T & B \\ C & D \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix}. \quad (3.26)$$

Proof. It is seen from the Stein identity (3.24) that $\mathcal{G} := \text{Ran} \begin{bmatrix} T \\ C \end{bmatrix}$ is a uniformly positive subspace of the Kreĭn space $\mathcal{K} = \mathcal{X} \oplus \mathcal{Y} \oplus \mathcal{U}$ with inner product induced by the operator $\begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix}$. The Kreĭn-space orthogonal projection of \mathcal{K} onto \mathcal{G} is given by $\mathcal{P}_{\mathcal{G}} = \begin{bmatrix} T \\ C \end{bmatrix} \begin{bmatrix} T \\ C \end{bmatrix}^{[*]}$ where the Kreĭn-space adjoint of $\begin{bmatrix} T \\ C \end{bmatrix}$ is given by

$$\begin{bmatrix} T \\ C \end{bmatrix}^{[*]} = P^{-1} \begin{bmatrix} T^* & C^* \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix}.$$

Therefore, the Kreĭn-space orthogonal projection $\mathcal{P}_{\mathcal{G}^{\perp}}$ equals

$$\mathcal{P}_{\mathcal{G}^{\perp}} = I_{\mathcal{K}} - \mathcal{P}_{\mathcal{G}} = I - \begin{bmatrix} T \\ C \end{bmatrix} P^{-1} \begin{bmatrix} T^* & C^* \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix}. \quad (3.27)$$

On the other hand, since \mathcal{G} is a uniformly positive subspace of \mathcal{K} , its orthogonal complement \mathcal{G}^{\perp} is also a Kreĭn space in inner product inherited from \mathcal{K} with inertia equal to that of J on $\mathcal{Y} \oplus \mathcal{U}$. Therefore there is an injective isometry

$$\begin{bmatrix} B \\ D \end{bmatrix} : \left(\begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}, J \right) \rightarrow \mathcal{K} \quad \text{such that} \quad \mathcal{P}_{\mathcal{G}^{\perp}} = \begin{bmatrix} B \\ D \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix}^{[*]}.$$

Since $\begin{bmatrix} B \\ D \end{bmatrix}^{[*]} = J \begin{bmatrix} B^* & D^* \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix}$, it follows that

$$\mathcal{P}_{\mathcal{G}^{\perp}} = \begin{bmatrix} B \\ D \end{bmatrix} J \begin{bmatrix} B^* & D^* \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix}. \quad (3.28)$$

Multiplying the two expressions (3.27) and (3.28) for $\mathcal{P}_{\mathcal{G}^{\perp}}$ by $\begin{bmatrix} P^{-1} & 0 \\ 0 & J \end{bmatrix}$ on the right and using the subsequent equation gives (3.25). Equality (3.26) then follows from the injectivity of $\begin{bmatrix} B \\ D \end{bmatrix}$. \square

With the operators B and D subject to operator equalities (3.25), (3.26) in hand, the next step is to let

$$\Theta(z) = D + zC(I - zT)^{-1}B \quad (3.29)$$

and then the identity (3.20) follows from (3.25) whereas the identity

$$\frac{J - \Theta(z)^* J \Theta(\zeta)}{1 - \bar{z}\zeta} = B^*(I - \bar{z}T^*)^{-1}P(I - \zeta T)^{-1}B$$

is a consequence of (3.26). The function Θ obtained this way also satisfies inequalities (3.21).

Remark 3.7. If a solution P to the Stein equation (3.7) is strictly positive definite, it then follows that the operator T is strongly stable, which in turn implies that the function Θ is J -inner, i.e., that is, the nontangential boundary values $\Theta(t)$ exist for almost all $t \in \mathbb{T}$ and are J -unitary: $\Theta(t)J\Theta(t)^* = J$.

Theorem 3.8. *Let P be a strictly positive solution to the Stein equation (3.24) and let Θ be a J -bicontractive function satisfying the identity (3.20). Then a function $S : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is a solution to the problem AIP if and only if it is of the form*

$$S = (\Theta_{11}\mathcal{E} + \Theta_{12})(\Theta_{21}\mathcal{E} + \Theta_{22})^{-1} \quad (3.30)$$

for some $\mathcal{E} \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$.

Proof. Substituting the block decomposition $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$ conformal with that of J into inequalities (3.23) gives in particular,

$$\begin{aligned} \Theta_{21}(z)\Theta_{21}(z)^* - \Theta_{22}(z)\Theta_{22}(z)^* &\leq -I_{\mathcal{U}}, \\ \Theta_{12}(z)^*\Theta_{12}(z) - \Theta_{22}(z)^*\Theta_{22}(z) &\leq -I_{\mathcal{U}}, \end{aligned}$$

from which it follows that $\Theta_{22}(z)$ is invertible and that $\|\Theta_{22}^{-1}(z)\Theta_{21}(z)\| \leq 1$ for all $z \in \mathbb{D}$. Therefore,

$$\Theta_{21}(z)\mathcal{E}(z) + \Theta_{22}(z) = \Theta_{22}(z)(\Theta_{22}^{-1}(z)\Theta_{21}(z)\mathcal{E}(z) + I)$$

is invertible for all $z \in \mathbb{D}$ and $\mathcal{E} \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ and thus the formula (3.30) makes sense.

One can now substitute (3.20) into (3.19) and conclude that the kernel \mathbb{K}_S is positive on $\mathbb{D} \times \mathbb{D}$ if and only if

$$\tilde{K}_S(z, \zeta) = \begin{bmatrix} I & -S(z) \end{bmatrix} \frac{\Theta(z)J\Theta(\zeta)^*}{1 - z\bar{\zeta}} \begin{bmatrix} I \\ -S(\zeta)^* \end{bmatrix} \succeq 0. \quad (3.31)$$

Set

$$u = \Theta_{11} - S\Theta_{21}, \quad v = S\Theta_{22} - \Theta_{12}. \quad (3.32)$$

Then $\begin{bmatrix} u & -v \end{bmatrix} := \begin{bmatrix} I & -S \end{bmatrix} \Theta$, then it follows that

$$\frac{u(z)u(\zeta)^* - v(z)v(\zeta)^*}{1 - \langle z, \zeta \rangle} \succeq 0,$$

which is equivalent, by Leech's theorem (see [RR1985, page 107]), to a factorization $v(z) = u(z)\mathcal{E}(z)$ for some $\mathcal{E} \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$. On account of (3.32), this in turn can be written as

$$S\Theta_{22} - \Theta_{12} = (\Theta_{11} - S\Theta_{21})\mathcal{E}.$$

The latter can be rearranged as $S(\Theta_{21}\mathcal{E} + \Theta) = \Theta_{11}\mathcal{E} + \Theta_{12}$ which in turn, is equivalent to (3.30). \square

The formal obstacle to the use of the parametrization (3.30) in case $P \geq 0$ is singular is the presence of P^{-1} in the formula (3.21) for Θ (the inverse of P also appears implicitly in formula (3.29) since the entries in this formula must satisfy equality (3.25)). A naive attempt to overcome this difficulty (in case $\dim \mathcal{X} < \infty$) would be to replace the inverse of P by its Moore-Penrose pseudoinverse. Not for the general **IP**, but at least for the left-tangential Nevanlinna-Pick problem (3.1), the formula (3.21) produces all solutions to the problem if the parameter \mathcal{E} is taken in the form

$$\mathcal{E}(z) = U \begin{bmatrix} \tilde{\mathcal{E}}(z) & 0 \\ 0 & I_\nu \end{bmatrix} V, \quad \text{where } \nu = \text{rank}(P + N^*N) - \text{rank} P,$$

where U and V are two matrices depending only on interpolation data and where $\tilde{\mathcal{E}}$ is an arbitrary Schur-class function. It was shown in [Du1984] for the matricial Schur-Carathéodory-Fejér problem and in [BD1998] for the general problem **IP** (still with $\dim \mathcal{X} < \infty$) that a similar result holds with an appropriate choice of the pseudoinverse of P (not the Moore-Penrose in general) satisfying certain invariance relations.

In the case $\dim \mathcal{X} = \infty$, this method does not seem to work beyond the situation where the compression of P to the orthogonal complement of its kernel is strictly positive definite. The following alternative approach handles the problem **AIP** regardless of whether the operator P is strictly positive definite or just positive semidefinite.

3.4. Redheffer parametrization of the solution set. Once again the starting point is the Stein identity (3.7) according to which

$$\|P^{\frac{1}{2}}x\|_{\mathcal{X}}^2 + \|Nx\|_{\mathcal{U}}^2 = \|P^{\frac{1}{2}}Tx\|_{\mathcal{X}}^2 + \|Ex\|_{\mathcal{Y}}^2 \quad \text{for all } x \in \mathcal{X}.$$

let $\mathcal{X}_0 = \overline{\text{Ran} P^{\frac{1}{2}}}$; the conclusion from the Stein equality then is that there exists a well defined isometry V with domain \mathcal{D}_V and range \mathcal{R}_V

equal to

$$\mathcal{D}_V = \overline{\text{Ran}} \begin{bmatrix} P^{\frac{1}{2}} \\ N \end{bmatrix} \subseteq \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{U} \end{bmatrix} \quad \text{and} \quad \mathcal{R}_V = \overline{\text{Ran}} \begin{bmatrix} P^{\frac{1}{2}}T \\ E \end{bmatrix} \subseteq \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{Y} \end{bmatrix},$$

respectively, which is uniquely determined by the identity

$$V \begin{bmatrix} P^{\frac{1}{2}}x \\ Nx \end{bmatrix} = \begin{bmatrix} P^{\frac{1}{2}}Tx \\ Ex \end{bmatrix} \quad \text{for all } x \in \mathcal{X}. \quad (3.33)$$

Let the defect spaces be defined by

$$\Delta := \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{U} \end{bmatrix} \ominus \mathcal{D}_V \quad \text{and} \quad \Delta_* := \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{Y} \end{bmatrix} \ominus \mathcal{R}_V \quad (3.34)$$

and let $\tilde{\Delta}$ and $\tilde{\Delta}_*$ denote isomorphic copies of Δ and Δ_* , respectively, with unitary identification maps

$$\mathbf{i} : \Delta \rightarrow \tilde{\Delta} \quad \text{and} \quad \mathbf{i}_* : \Delta_* \rightarrow \tilde{\Delta}_*.$$

With these identification maps let us define a unitary colligation matrix \mathbf{U} from $\mathcal{D}_V \oplus \Delta \oplus \tilde{\Delta}_* = \mathcal{X} \oplus \mathcal{U} \oplus \tilde{\Delta}_*$ onto $\mathcal{R}_V \oplus \Delta_* \oplus \tilde{\Delta} = \mathcal{X} \oplus \mathcal{Y} \oplus \tilde{\Delta}$ by

$$\mathbf{U} = \begin{bmatrix} V & 0 & 0 \\ 0 & 0 & \mathbf{i}_*^* \\ 0 & \mathbf{i} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{D}_V \\ \Delta \\ \tilde{\Delta}_* \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}_V \\ \Delta_* \\ \tilde{\Delta} \end{bmatrix}, \quad (3.35)$$

which will be also decomposed as

$$\mathbf{U} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{U} \\ \tilde{\Delta}_* \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{Y} \\ \tilde{\Delta} \end{bmatrix}. \quad (3.36)$$

Write Σ for the characteristic function associated with this colligation \mathbf{U} , i.e.,

$$\Sigma(z) = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{bmatrix} + z \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (I - zA)^{-1} \begin{bmatrix} B_1 & B_2 \end{bmatrix} \quad (z \in \mathbb{D}), \quad (3.37)$$

and decompose Σ as

$$\Sigma(z) = \begin{bmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \tilde{\Delta}_* \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{Y} \\ \tilde{\Delta} \end{bmatrix}. \quad (3.38)$$

A straightforward calculation based on the fact that \mathbf{U} is coisometric gives

$$\frac{I - \Sigma(z)\Sigma(\zeta)^*}{1 - z\bar{\zeta}} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (I - zA)^{-1} (I - \bar{\zeta}A^*)^{-1} \begin{bmatrix} C_1^* & C_2^* \end{bmatrix}, \quad (3.39)$$

which implies in particular that Σ belongs to the Schur class $\mathcal{S}(\mathcal{U} \oplus \tilde{\Delta}_*, \mathcal{Y} \oplus \tilde{\Delta})$.

Theorem 3.9. *A function $S : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is a solution of the problem **AIP** if and only if*

$$S = \mathcal{R}_\Sigma[\mathcal{E}] := \Sigma_{11} + \Sigma_{12}(I - \mathcal{E}\Sigma_{22})^{-1}\mathcal{E}\Sigma_{21} \quad (3.40)$$

for some $\mathcal{E} \in \mathcal{S}(\tilde{\Delta}, \tilde{\Delta}_*)$.

Note that by construction, $\Sigma_{22}(0) = 0$ so that formula (3.40) makes sense for any Schur-class function $\mathcal{E} \in \mathcal{S}(\tilde{\Delta}, \tilde{\Delta}_*)$. The proof of Theorem 3.9 can be found in [KKY1987, Kh1998].

In more detail, it is not hard to see that if \mathcal{K} is a Hilbert space containing \mathcal{X} and

$$U = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{K} \\ \mathcal{U} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{K} \\ \mathcal{Y} \end{bmatrix} \quad (3.41)$$

is a unitary operator such that

$$U \begin{bmatrix} P^{\frac{1}{2}}x \\ Nx \end{bmatrix} = \begin{bmatrix} P^{\frac{1}{2}}Tx \\ Ex \end{bmatrix} \quad \text{for all } x \in \mathcal{X},$$

(i.e., U is a unitary extension of the isometry V (3.33)), then the characteristic function

$$S(z) = A_{22} + zA_{12}(I - zA_{11})^{-1}A_{21}$$

is a solution of the problem **AIP**. A much less trivial fact (established in [KKY1987]) is that *any* solution to the problem **AIP** arises in this way. Then it remains to parametrize all unitary extensions U of the form (3.41) of the isometry (3.33) or (which is even better) to parametrize the set of characteristic functions of all such extensions. The latter was done in [ArG1983, ArG1992] via coupling of unitary colligations.

The conclusion of this section is a result needed in the sequel; proofs can be found in [BBtH2011b].

Proposition 3.10. *Let Σ be the Schur-class function constructed as in (3.37) and decomposed as in (3.38), and let S be of the form (3.40) for a given $\mathcal{E} \in \mathcal{S}(\tilde{\Delta}, \tilde{\Delta}_*)$. Then the de Branges-Rovnyak kernels K_S and $K_{\mathcal{E}}$ (see (2.10)) are related as follows:*

$$K_S(z, \zeta) = G(z)K_{\mathcal{E}}(z, \zeta)G(\zeta)^* + \Gamma(z)\Gamma(\zeta)^* \quad (3.42)$$

where the functions G and Γ are defined on \mathbb{D} in terms of Σ by

$$\begin{aligned} G(z) &= \Sigma_{12}(z)(I - \mathcal{E}(z)\Sigma_{22}(z))^{-1}, \\ \Gamma(z) &= (C_1 + G(z)\mathcal{E}(z)C_2)(I - zA)^{-1}. \end{aligned} \quad (3.43)$$

Furthermore, the following equality holds for all $z \in \mathbb{D}$:

$$\Gamma(z)P^{\frac{1}{2}} = (E - S(z)N(I - zT)^{-1} = F^S(z). \quad (3.44)$$

4. INTERPOLATION IN $\mathcal{H}(S)$

Interpolation problems in de Branges-Rovnyak spaces have not been considered until recently. The lack of interest in this topic can be explained by the fact that Hilbert space interpolation is well understood and no surprises are expected. However the results arising from the general Hilbert-space structure can be made much more explicit and concrete for this particular setting, as is discussed below. Much of this Section is based on the papers of Ball, Bolotnikov, and ter Horst [BB2008, BBtH2011a, BBtH2011b].

Throughout this section, the Schur-class function $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ is fixed. Hence the space $\mathcal{H}(S)$ consists of \mathcal{Y} -valued functions and every function $f \in \mathcal{H}(S)$ induces the multiplication operator $M_f : \mathbb{C} \rightarrow \mathcal{H}(S)$ by the formula $M_f \alpha = f(z)\alpha$. In what follows, A^* will denote the adjoint of $A : \mathcal{X} \rightarrow \mathcal{H}(S) \subset H^2(\mathcal{Y})$ in the metric of $H^2(\mathcal{Y})$, and $A^{[*]}$ will denote the adjoint of A in the metric of $\mathcal{H}(S)$. Since these metrics are different (unless S is inner), the adjoints A^* and $A^{[*]}$ are not equal in general.

As in Section 3, the starting point is a simple left-tangential Nevanlinna-Pick problem: *Given n distinct points $z_1, \dots, z_n \in \mathbb{D}$ and given vectors $E_1, \dots, E_n \in \mathcal{Y}$ and given complex numbers y_1, \dots, y_n , find a function $f \in \mathcal{H}(S)$ such that*

$$f(z_i)^* E_i = y_i \quad \text{for } i = 1, \dots, n. \quad (4.1)$$

Making use of formula (2.3) one may write the left hand side expression in (4.1) in terms of the adjoint operators $M_f^* : H^2(\mathcal{Y}) \rightarrow \mathbb{C}$ and $M_f^{[*]} : \mathcal{H}(S) \rightarrow \mathbb{C}$. Indeed,

$$f(z_i)^* E_i = M_f^* E_i k_{S_z}(\cdot, z_i) = M_f^* (E_i (1 - z \bar{z}_i)^{-1}) \quad \text{for } i = 1, \dots, n, \quad (4.2)$$

and on the other hand,

$$\begin{aligned} f(z_i)^* E_i &= M_f^{[*]} (K_S(\cdot, z_i) E_i) \\ &= M_f^{[*]} ((E_i - S S(z_i)^* E_i) k_{S_z}(\cdot, z_i)) \\ &= M_f^{[*]} ((E_i - S N_i) k_{S_z}(\cdot, z_i)) \quad \text{for } i = 1, \dots, n, \end{aligned} \quad (4.3)$$

where $N_i := S(z_i)^* E_i$ (recall that the function S is given). Making use of matrices (3.3) and letting $\mathbf{y} = [y_1 \ \dots \ y_n]$, one may rewrite n conditions in (4.2) and (4.3) as

$$M_f^* : E(I - zT)^{-1} x \mapsto \mathbf{y}x \quad \text{and} \quad M_f^{[*]} : (E - S(z)N)(I - zT)^{-1} x \mapsto \mathbf{y}x.$$

respectively, holding for all $x \in \mathbb{C}^n$. Next use the observability operators (3.4) and the operator (3.11) to write the latter equalities in more compact form

$$M_f^* \mathcal{O}_{E,T} x = \mathbf{y}x \quad \text{and} \quad M_f^{[*]} F^S x = \mathbf{y}x$$

or equivalently,

$$\mathcal{O}_{E,T}^* f = \mathbf{y}^* \quad \text{and} \quad (F^S)^{[*]} f = \mathbf{y}^*. \quad (4.4)$$

As will be shown below, the two latter conditions are equivalent in a much more general situation. The first condition in (3.4) looks very much the same as that in (3.10), and this condition will be used to formulate the problem **IP** for functions in the space $\mathcal{H}(S)$.

IP _{$\mathcal{H}(S)$} : Given a Schur-class function $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, given an output-stable pair (E, T) of operators $E \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $T \in \mathcal{L}(\mathcal{X})$, and given a functional $\mathbf{y} \in \mathcal{L}(\mathcal{X}, \mathbb{C})$, find a function $f \in \mathcal{H}(S)$ such that $\mathcal{O}_{E,T}^* f = \mathbf{y}^*$ and $\|f\|_{\mathcal{H}(S)} \leq 1$.

With the data set as above, one can introduce the operator $N \in \mathcal{L}(\mathcal{X}, \mathcal{U})$ via formula (3.10), that is, via its adjoint

$$N^* u = \mathcal{O}_{E,T}^* (Su), \quad u \in \mathcal{U}. \quad (4.5)$$

Since, S is a Schur-class function, the pair (N, T) is output-stable, and the operator F^S given by (3.11) maps \mathcal{X} into $\mathcal{H}(S)$. Since S trivially solves the problem **IP** with the current choice of N , inequality (3.9) holds by Theorem 3.1 while equality (3.12) holds by Theorem 3.3. Equality (3.12) can be written in the operator form as

$$P = (F^S)^{[*]} F^S. \quad (4.6)$$

Finally the equalities

$$\begin{aligned} \langle (F^S)^{[*]} f, x \rangle_{\mathcal{X}} &= \langle f, F^S x \rangle_{\mathcal{H}(S)} = \langle f, (I - T_S T_S^*) \mathcal{O}_{E,T} x \rangle_{\mathcal{H}(S)} \\ &= \langle f, \mathcal{O}_{E,T} x \rangle_{H^2(\mathcal{Y})} = \langle \mathcal{O}_{E,T}^* f, x \rangle_{\mathcal{X}} \end{aligned}$$

hold for all $f \in \mathcal{H}(S)$ and $x \in \mathcal{X}$. Therefore, $(F^S)^{[*]} = \mathcal{O}_{E,T}^*|_{\mathcal{H}(S)}$ and conditions (4.4) are equivalent in the general setting of the problem **IP**.

As in the Schur-class setting, boundary interpolation problems cannot be embedded into the framework of the problem **IP**. To handle the boundary case, the stability assumption on the pair (E, T) need be relaxed. If the pair (E, T) is not output-stable, we cannot use formula (4.5) to define N . Thus, the operator N must be a part of interpolation data. Also the interpolation condition $\mathcal{O}_{E,T}^* f = \mathbf{y}^*$ cannot be formulated in this form since $\mathcal{O}_{E,T}$ does not map \mathcal{X} into $H^2(\mathcal{Y})$ and thus its

range is not in $\mathcal{H}(S) \subset H^2(\mathcal{Y})$. Instead, one can assume that given $S(z)$, E , N , T are such that the operator F^S defined as in (3.11) maps \mathcal{X} into $\mathcal{H}(S)$. Under this assumption one may use the second formula in (4.4) as the interpolation condition; on the other hand, P can be defined via formula (4.5) instead of (3.6). For the reasons already clear from what was seen in the previous section, it makes sense to assume that the Stein identity (3.7) is in force.

Definition 4.1. The data set

$$\mathcal{D} = \{S, T, E, N, \mathbf{y}\} \quad (4.7)$$

consisting of a Schur-class function $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ and operators $T \in \mathcal{L}(\mathcal{X})$, $E \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $N \in \mathcal{L}(\mathcal{X}, \mathcal{U})$, and $\mathbf{y} \in \mathcal{L}(\mathcal{X}, \mathbb{C})$. is said to be **AIP** $_{\mathcal{H}(S)}$ -admissible if:

- (1) The function $\begin{bmatrix} E \\ N \end{bmatrix} (I - zT)^{-1}x$ is holomorphic on \mathbb{D} for each $x \in \mathcal{X}$.
- (2) The operator F^S (3.11) maps \mathcal{X} into $\mathcal{H}(S)$.
- (3) The operator $P := (F^S)^{[*]}F^S$ satisfies the Stein equation (3.7).

These preparations lead to the formulation of the problem **AIP** $_{\mathcal{H}(S)}$:

*Given an **AIP** $_{\mathcal{H}(S)}$ -admissible data set (4.7), find all $f \in \mathcal{H}(S)$ such that*

$$M_{F^S}^{[*]}f = \mathbf{y}^* \quad \text{and} \quad \|f\|_{\mathcal{H}(S)} \leq 1. \quad (4.8)$$

The discussion preceding Definition 4.1 shows that the problem **IP** $_{\mathcal{H}(S)}$ is a particular case of the problem **AIP** $_{\mathcal{H}(S)}$.

4.1. The problem **AIP $_{\mathcal{H}(S)}$ as a linear operator equation.** Consider the following operator interpolation problem with norm constraint: *Given Hilbert space operators $A \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$, find all $X \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ that satisfy the conditions*

$$AX = B \quad \text{and} \quad \|X\| \leq 1. \quad (4.9)$$

According to the Douglas lemma [D1965], there is an $X \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ satisfying (4.9) if and only if $AA^* \geq BB^*$. If this is the case, then there exist (unique) contractions $X_1 \in \mathcal{L}(\mathcal{U}, \overline{\text{Ran}}A)$ and $X_2 \in \mathcal{L}(\mathcal{Y}, \overline{\text{Ran}}A)$ such that

$$(AA^*)^{\frac{1}{2}}X_1 = B, \quad (AA^*)^{\frac{1}{2}}X_2 = A, \quad \text{Ker}X_1 = \text{Ker}B, \quad \text{Ker}X_2 = \text{Ker}A. \quad (4.10)$$

The next characterization of all operators X subject to (4.9) can be found in [BBtH2011b].

Lemma 4.2. *Assume $AA^* \geq BB^*$ and let $X \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. Then the following statements are equivalent:*

- (1) *X satisfies conditions (4.9).*
- (2) *The operator*

$$\begin{bmatrix} I_{\mathcal{H}_1} & B^* & X^* \\ B & AA^* & A \\ X & A^* & I_{\mathcal{H}_2} \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \quad (4.11)$$

is positive semidefinite.

- (3) *X is of the form*

$$X = X_2^* X_1 + (I - X_2^* X_2)^{\frac{1}{2}} K (I - X_1^* X_1)^{\frac{1}{2}} \quad (4.12)$$

where X_1 and X_2 are defined as in (4.10) and where the parameter K is an arbitrary contraction from $\overline{\text{Ran}}(I - X_1^ X_1)$ into $\overline{\text{Ran}}(I - X_2^* X_2)$.*

Moreover, if X satisfies (4.9), then X is unique if and only if X_1 is isometric on \mathcal{U} or X_2 is isometric on \mathcal{Y} .

Remark 4.3. It follows from (4.12) that there is a unique X subject to conditions (4.9) if and only if X_1 is isometric on \mathcal{U} or X_2 is isometric on \mathcal{Y} . Furthermore, since X_2 is a coisometry, it follows that $(I - X_2^* X_2)^{\frac{1}{2}}$ is the orthogonal projection onto $\mathcal{U} \ominus \text{Ker} A = \mathcal{U} \ominus \text{Ker} X_1$. This implies that for each K in (4.12) and each $u \in \mathcal{U}$,

$$\|Xu\|^2 = \|X_2^* X_1 u\|^2 + \|(I - X_2^* X_2)^{\frac{1}{2}} K (I - X_1^* X_1)^{\frac{1}{2}} u\|^2,$$

so that $X_2^* X_1$ is the minimal norm solution to the problem (4.9) (see [BBtH2011b]).

Upon specifying the preceding discussion to the case where

$$A = (F^S)^{[*]} : \mathcal{H}(K_S) \rightarrow \mathcal{X}, \quad B = \mathbf{y}^* \in \mathcal{X} \cong \mathcal{L}(\mathbb{C}, \mathcal{X}), \quad (4.13)$$

then it is readily seen that solutions $X : \mathbb{C} \rightarrow \mathcal{H}(S)$ to problem (4.9) necessarily have the form of a multiplication operator M_f for some function $f \in \mathcal{H}(S)$. This observation leads to the following solvability criterion.

Theorem 4.4. *The problem $\mathbf{AIP}_{\mathcal{H}(S)}$ has a solution if and only if*

$$P \geq \mathbf{y}^* \mathbf{y}, \quad \text{where } P := (F^S)^{[*]} F^S. \quad (4.14)$$

Assuming for simplicity that the operator P is strictly positive definite, it is readily seen that

$$X_1 = P^{-\frac{1}{2}} \mathbf{y}^* \in \mathcal{X}, \quad X_2 = P^{-\frac{1}{2}} (F^S)^{[*]} \in \mathcal{L}(\mathcal{H}(S), \mathcal{X})$$

are the operators X_1 and X_2 from (4.10) after specialization to the case (4.13). The conclusion from (4.12) is that all solutions f to the problem $\mathbf{AIP}_{\mathcal{H}(S)}$ are given by the formula

$$f = F^S P^{-1} \mathbf{y}^* + \sqrt{1 - \|P^{-\frac{1}{2}} \mathbf{y}^*\|^2} \cdot (I - F^S P^{-1} (F^S)^{[*]})^{\frac{1}{2}} K \quad (4.15)$$

where K is a function from the unit ball of the space $\text{Ran}(I - F^S P^{-1} (F^S)^{[*]})^{\frac{1}{2}}$. The latter space is in fact the reproducing kernel Hilbert space with reproducing kernel

$$\tilde{K}_S(z, \zeta) = K_S(z, \zeta) - F^S(z) P^{-1} F^S(\zeta)^* \quad (4.16)$$

and the second term on the right side of (4.15) is nothing else but a function $h \in \mathcal{H}(\tilde{K}_S)$ such that

$$\|h\|_{\mathcal{H}(\tilde{K}_S)} \leq \sqrt{1 - \|P^{-\frac{1}{2}} \mathbf{y}^*\|^2}. \quad (4.17)$$

Theorem 4.5. *Assume that condition (4.14) holds and that P is strictly positive definite. Let \tilde{K}_S be the kernel defined in (4.16). Then all solutions f to the problem $\mathbf{AIP}_{\mathcal{H}(S)}$ are described by the formula*

$$f(z) = F^S(z) P^{-1} \mathbf{y}^* + h(z) \quad (4.18)$$

where h is a free parameter from $\mathcal{H}(\tilde{K}_S)$ subject to norm constraint (4.17). The problem $\mathbf{AIP}_{\mathcal{H}(S)}$ has a unique solution if and only if $\|P^{-\frac{1}{2}} \mathbf{y}^*\| = 1$ or $\tilde{K}_S(z, \zeta) \equiv 0$.

Remark 4.6. The function h on the right hand side of (4.18) represents in fact the general solution of the homogeneous interpolation problem (with interpolation condition $(F^S)^{[*]} f = 0$). If h runs through the whole space $\mathcal{H}(\tilde{K}_S)$, then formula (4.18) produces all functions $f \in \mathcal{H}(S)$ such that $(F^S)^{[*]} f = \mathbf{y}^*$. This unconstrained interpolation problem has a solution if and only if $\mathbf{y}^* \in \text{Ran } P^{\frac{1}{2}}$ and has a unique solution if and only if $\tilde{K}_S(z, \zeta) \equiv 0$.

Parametrization of the form (4.18) is typical for interpolation problems in reproducing kernel Hilbert spaces. The most interesting part in this topic is to get a more detailed characterization of all solutions of the homogeneous problem. For the case $S \equiv 0$, such a characterization is given by Beurling-Lax theorem. It is quite remarkable that an analog of the Beurling-Lax theorem holds in general de Branges-Rovnyak space. For getting these analogs the assumption (3) in Definition 4.1 (which has not been used so far) is crucial.

4.2. Analytic descriptions of the solution set. Following the strategy from Section 3, the starting point is the analog of Theorem 3.3 for the current Hilbert space setting.

Theorem 4.7. *A function $f : \mathbb{D} \rightarrow \mathcal{Y}$ is a solution of the problem $\mathbf{AIP}_{\mathcal{H}(S)}$ with data set (4.7) if and only if the kernel*

$$\mathbf{K}(z, \zeta) = \begin{bmatrix} 1 & \mathbf{y} & f(\zeta)^* \\ \mathbf{y}^* & P & F^S(\zeta)^* \\ f(z) & F^S(z) & K_S(z, \zeta) \end{bmatrix} \quad (z, \zeta \in \mathbb{D}), \quad (4.19)$$

is positive on $\mathbb{D} \times \mathbb{D}$. Here P , F^S and K_S are given by (4.6), (3.17) and (2.10), respectively.

Proof. By Lemma 4.2 specialized to A and B as in (4.13) and $X = M_f$, One can now conclude that f is a solution to the problem $\mathbf{AIP}_{\mathcal{H}(S)}$ (that is, it meets conditions (4.8)) if and only if the following operator is positive semidefinite:

$$\mathbf{P} := \begin{bmatrix} 1 & \mathbf{y} & M_f^{[*]} \\ \mathbf{y}^* & (F^S)^{[*]} F^S & (F^S)^{[*]} \\ M_f & F^S & I_{\mathcal{H}(S)} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{y} & M_f^{[*]} \\ \mathbf{y}^* & P & (F^S)^{[*]} \\ M_f & F^S & I_{\mathcal{H}(S)} \end{bmatrix} \geq 0.$$

As in the proof of Theorem 3.3, it is useful to observe that for every $g \in \mathbb{C} \oplus \mathcal{X} \oplus \mathcal{H}(S)$ of the form

$$g(z) = \sum_{j=1}^r \begin{bmatrix} c_j \\ x_j \\ K_S(\cdot, z_j) y_j \end{bmatrix} \quad (c_j \in \mathbb{C}, y_j \in \mathcal{Y}, x_j \in \mathcal{X}, z_j \in \mathbb{D}) \quad (4.20)$$

the identity

$$\langle \mathbf{P}g, g \rangle_{\mathbb{C} \oplus \mathcal{X} \oplus \mathcal{H}(S)} = \sum_{j, \ell=1}^r \left\langle \mathbf{K}(z_j, z_\ell) \begin{bmatrix} c_\ell \\ x_\ell \\ y_\ell \end{bmatrix}, \begin{bmatrix} c_j \\ x_j \\ y_j \end{bmatrix} \right\rangle_{\mathbb{C} \oplus \mathcal{X} \oplus \mathcal{Y}} \quad (4.21)$$

holds. Since the set of vectors of the form (4.20) is dense in $\mathbb{C} \oplus \mathcal{X} \oplus \mathcal{H}(S)$, the identity (4.21) now implies that the operator \mathbf{P} is positive semidefinite if and only if the quadratic form on the right hand side of (4.21) is nonnegative, i.e., if and only if the kernel (4.19) is positive on $\mathbb{D} \times \mathbb{D}$. \square

The next observation is that for any $\mathbf{AIP}_{\mathcal{H}(S)}$ -admissible data set (4.7), the Schur-class function S is a solution of the Schur-class problem \mathbf{AIP} with the data set $\{T, E, N, P = (F^S)^{[*]} F^S\}$. If P is strictly positive definite, then by Theorem 3.8, S is necessarily of the form (3.30) for a J -inner function Θ explicitly constructed from the data set

and a Schur-class function $\mathcal{E} \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ which is recovered from S by the formula

$$\mathcal{E} = (\Theta_{11} - S\Theta_{21})^{-1}(S\Theta_{22} - \Theta_{12}). \quad (4.22)$$

Furthermore, the formula (3.31) for the kernel \tilde{K}_S can be written in terms of this \mathcal{E} as

$$\tilde{K}_S(z, \zeta) = u(z)K_{\mathcal{E}}(z, \zeta)u(\zeta)^*, \quad (4.23)$$

where

$$K_{\mathcal{E}}(z, \zeta) = \frac{I_{\mathcal{Y}} - \mathcal{E}(z)\mathcal{E}(\zeta)^*}{1 - z\bar{\zeta}}, \quad u(z) = \Theta_{11}(z) - S(z)\Theta_{21}(z). \quad (4.24)$$

Theorem 4.8. *Assume that the data set of the problem $\mathbf{AIP}_{\mathcal{H}(S)}$ is such that the operator $P = (F^S)^{[*]}F^S$ is strictly positive definite. Let $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$ be a J -inner function satisfying (3.22) and let $\mathcal{E} \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$, u and F^S be given as in (4.22) (4.24) and (3.17). Then:*

- (1) *All solutions f of the problem $\mathbf{AIP}_{\mathcal{H}(S)}$ are parametrized by the formula*

$$f = F^S P^{-1} \mathbf{y}^* + uh, \quad (4.25)$$

where h is a function from the de Branges-Rovnyak space $\mathcal{H}(\mathcal{E})$ such that

$$\|h\|_{\mathcal{H}(\mathcal{E})} \leq \sqrt{1 - \|P^{-\frac{1}{2}} \mathbf{y}^*\|^2}. \quad (4.26)$$

- (2) *Representation (4.25) is orthogonal in the metric of $\mathcal{H}(S)$.*
- (3) *The multiplication operator $M_u : \mathcal{H}(\mathcal{E}) \rightarrow \mathcal{H}(S)$ is isometric.*
- (4) *For f defined by (4.25),*

$$\|f\|_{\mathcal{H}(S)}^2 = \|P^{-\frac{1}{2}} \mathbf{y}^*\|_{\mathcal{X}}^2 + \|h\|_{\mathcal{H}(\mathcal{E})}^2. \quad (4.27)$$

Proof. In the case P is strictly positive definite, one can take its Schur complement in \mathbf{K} to get, on account of (4.23), the equivalent inequality

$$\begin{bmatrix} 1 - \|P^{-\frac{1}{2}} \mathbf{y}^*\|^2 & f(\zeta)^* - \mathbf{y} P^{-1} F^S(\zeta)^* \\ f(z) - F^S(z) P^{-1} \mathbf{y}^* & u(z) K_{\mathcal{E}}(z, \zeta) u(\zeta)^* \end{bmatrix} \succeq 0. \quad (4.28)$$

The latter positivity is equivalent to the function $f - F^S P^{-1} \mathbf{y}^*$ be of the form uh for some $h \in \mathcal{H}(\mathcal{E})$ subject to norm constraint (4.26). Statement (2) follows by Remark 4.3 and the isometric property of the operator $M_u : \mathcal{H}(\mathcal{E}) \rightarrow \mathcal{H}(S)$ is a consequence of factorization (4.23). The last statement follows from parts (2) and (3) and the fact that $\|F^S P^{-1} \mathbf{y}^*\|_{\mathcal{H}(S)}^2 = \|P^{-\frac{1}{2}} \mathbf{y}^*\|_{\mathcal{X}}^2$. \square

4.3. Description based on the Redheffer transform. Theorem 4.7 holds true even if P is not strictly positive definite. However, in this case one should use the Redheffer representation (3.40) for S rather than (3.30). Due to condition (4.14), there exists a (unique) $\tilde{\mathbf{y}}^* \in \mathcal{X} \ominus \text{Ker} P$ such that $\mathbf{y}^* = P^{\frac{1}{2}} \tilde{\mathbf{y}}^*$. As in the nondegenerate case, S is a solution of the Schur-class problem **AIP** with the data set $\{T, E, N, P = (F^S)^{[*]} F^S\}$ and therefore, it is of the form (3.40) for some (perhaps, not uniquely determined) Schur-class function \mathcal{E} . Nevertheless, identities (3.42) and (3.44) hold for functions G and Γ defined via formulas (3.43), and making use of these identities the kernel (4.19) can be written as

$$\mathbf{K}(z, \zeta) = \begin{bmatrix} 1 & \tilde{\mathbf{y}} P^{\frac{1}{2}} & f(\zeta)^* \\ P^{\frac{1}{2}} \tilde{\mathbf{y}}^* & P & P^{\frac{1}{2}} \Gamma(\zeta)^* \\ f(z) & \Gamma(z) P^{\frac{1}{2}} & G(z) K_{\mathcal{E}}(z, \zeta) G(\zeta)^* + \Gamma(z) \Gamma(\zeta)^* \end{bmatrix}.$$

The positivity of the latter kernel is equivalent to positivity of the Schur complement of P with respect to $\mathbf{K}(z, \zeta)$, that is, to the condition

$$\begin{bmatrix} 1 - \|\tilde{\mathbf{y}}\|^2 & f(\zeta)^* - \tilde{\mathbf{y}} \Gamma(\zeta)^* \\ f(z) - \Gamma(z) \tilde{\mathbf{y}}^* & G(z) K_{\mathcal{E}}(z, \zeta) G(\zeta)^* \end{bmatrix} \succeq 0 \quad (z, \zeta \in \mathbb{D}). \quad (4.29)$$

It follows from the identity (3.42) that the multiplication operators $M_G : h \mapsto Gh$ and $M_{\Gamma} : x \rightarrow \Gamma x$ are contractions from $\mathcal{H}(\mathcal{E})$ to $\mathcal{H}(S)$ and from $\mathcal{X}_0 = \overline{\text{Ran}} P^{\frac{1}{2}}$, respectively, and that the operator

$$\begin{bmatrix} M_G & M_{\Gamma} \end{bmatrix} : \begin{bmatrix} \mathcal{H}(K_{\mathcal{E}}) \\ \mathcal{X}_0 \end{bmatrix} \rightarrow \mathcal{H}(K_S)$$

is coisometric. Furthermore, since in the current case $P = (F^S)^{[*]} F^S$, it follows from (3.42) and (3.44) that M_{Γ} is an isometry and M_G a partial isometry. This leads to the following analog of Theorem 4.8.

Theorem 4.9. *All solutions f of the problem **AIP** $_{\mathcal{H}(S)}$ is given by the formula*

$$f(z) = \Gamma(z) \tilde{\mathbf{y}}^* + G(z) h(z) \quad (4.30)$$

with parameter h in $\mathcal{H}(\mathcal{E})$ subject to $\|h\|_{\mathcal{H}(\mathcal{E})} \leq \sqrt{1 - \|\tilde{\mathbf{y}}\|^2}$. Furthermore, for f defined by (4.30)

$$\|f\|_{\mathcal{H}(S)}^2 = \|\Gamma \tilde{\mathbf{y}}^*\|_{\mathcal{H}(S)}^2 + \|Gh\|_{\mathcal{H}(S)}^2 = \|\tilde{\mathbf{y}}\|_{\mathcal{X}}^2 + \|P_{\mathcal{H}(\mathcal{E}) \ominus \text{Ker} M_G} h\|_{\mathcal{H}(\mathcal{E})}^2 \quad (4.31)$$

and hence $f_{\min}(z) = \Gamma(z) \tilde{\mathbf{y}}^*$ is the unique minimal-norm solution.

The latter theorem is not a complete analog of Theorem 4.8 since (1) the Schur-class function \mathcal{E} is not determined uniquely and (2) the multiplication operator M_G is not isometric. To get a closer analog of

Theorem 4.8, it makes sense to assume that the operator T meets the condition

$$\left(\bigcap_{k \geq 1} \text{Ran}(T^*)^k \right) \cap \text{Ker} T^* = \{0\} \quad (4.32)$$

which is indeed satisfied for the following important particular cases:

- (1) T^* is injective (so $\text{Ker} T^* = \{0\}$),
- (2) T^* is nilpotent (so $\bigcap_{k \geq 1} \text{Ran}(T^*)^k = \{0\}$), and
- (3) $\dim \mathcal{X} < \infty$, or, more generally e.g., $T = \lambda I + K$ with $0 \neq \lambda \in \mathbb{C}$ and K compact (so $\mathcal{X} = \text{Ran}(T^*)^p \dot{+} \text{Ker}(T^*)^p$ once p is sufficiently large).

Theorem 4.10. *Let Σ be the Schur-class function defined in (3.37) from the $\mathbf{AIP}_{\mathcal{H}(S)}$ -admissible data set (4.7) and $P = (F^S)^{[*]} F^S$. If T satisfies condition (4.32), then*

- (1) *The Redheffer transform $\mathcal{R}_\Sigma : \mathcal{S}(\tilde{\Delta}, \tilde{\Delta}_*) \rightarrow \mathcal{S}(\mathcal{U}, \mathcal{Y})$ defined by formula (3.40) is injective.*
- (2) *For any $\mathcal{E} \in \mathcal{S}(\tilde{\Delta}, \tilde{\Delta}_*)$, the multiplication operator $M_G : f \rightarrow Gf$ by the function $G = \Sigma_{12}(I - \mathcal{E}\Sigma_{22})^{-1}$ is an isometry from $\mathcal{H}(\mathcal{E}) \rightarrow \mathcal{H}(S)$.*

Thus, if the operator T in the $\mathbf{AIP}_{\mathcal{H}(S)}$ -admissible data set (4.7) satisfies condition (4.32) then the Schur-class function \mathcal{E} in Theorem 4.9 is determined uniquely from the data set and Σ , and the formula (4.31) takes a simpler form

$$\|f\|_{\mathcal{H}(S)}^2 = \|\tilde{\mathcal{Y}}\|_{\mathcal{X}}^2 + \|h\|_{\mathcal{H}(\mathcal{E})}^2.$$

5. BOUNDARY BEHAVIOR AND BOUNDARY INTERPOLATION IN $\mathcal{H}(S)$

In this section, relations between boundary regularity of a Schur-class function S and boundary regularity of functions in the associated de Branges-Rovnyak space $\mathcal{H}(S)$ will be discussed. The next theorem presents a result of this type.

Theorem 5.1. *Let I be an open arc of \mathbb{T} , let S be a scalar-valued Schur-class function and let $A_S = R_0|_{\mathcal{H}(S)}$ be the model operator for S (see (2.17)). The following are equivalent:*

- (1) *S admits an analytic continuation across I and $|S(\zeta)| = 1$ for all $\zeta \in I$.*
- (2) *I is contained in the resolvent set of A_S^* .*
- (3) *Any function $f \in \mathcal{H}(S)$ admits an analytic continuation across I .*

For the proof the reader is referred to [FM2008] and to earlier sources [He1964, Chapter VIII] (for the case where S is inner) and [S1994, Chapter 5] (for the case where S is subject to the condition $\int_{\mathbb{T}} \log(1 - |S(\zeta)|) d\zeta = -\infty$).

The single-point (local) version of Theorem 5.1 is the following:

Theorem 5.2. *Let S be a scalar-valued Schur-class function and let $t_0 \in \mathbb{T}$. The following are equivalent:*

- (1) S admits an analytic continuation into a neighborhood \mathcal{U} of t_0 and is unimodular on $\mathcal{U} \cap \mathbb{T}$.
- (2) The operator $t_0 I - A_S^*$ is invertible on $\mathcal{H}(S)$.
- (3) Any function $f \in \mathcal{H}(S)$ admits an analytic continuation into \mathcal{U} .

Our next aim is to find conditions (in terms of S) guaranteeing a weaker but more natural property: the existence of the nontangential boundary limit

$$f(t_0) := \angle \lim_{z \rightarrow t_0} f(z) \quad (5.1)$$

(i.e., t_0 is approached from within an arbitrary but fixed Stolz angle with the vertex at t_0) for any function $f \in \mathcal{H}(S)$. Upon comparing statements (1) and (3) in Theorems 5.1 and 5.2, one may think that the desired condition might be that S admits a unimodular boundary limit $S(t_0)$. This condition is indeed necessary but not sufficient as the next theorem shows.

Theorem 5.3. *Let $S \in \mathcal{S}$ and $t_0 \in \mathbb{T}$. The following are equivalent:*

- (1) The boundary limit (5.1) exists for every $f \in \mathcal{H}(S)$.
- (2) S meets the Carathéodory-Julia condition

$$\liminf_{z \rightarrow t_0} \frac{1 - |S(z)|^2}{1 - |z|^2} < \infty \quad (5.2)$$

where z tends to t_0 unrestrictedly in \mathbb{D} .

- (3) The boundary limits

$$S_0 = \angle \lim_{z \rightarrow t_0} S(z) \quad \text{and} \quad S_1 = \angle \lim_{z \rightarrow t_0} S'(z) \quad (5.3)$$

exist and are subject to conditions

$$|S_0| = 1 \quad \text{and} \quad t_0 S_1 \overline{S_0} \in \mathbb{R}. \quad (5.4)$$

- (4) The boundary limit S_0 in (5.3) exists and the function $K_{t_0}(z) := \frac{1 - S(z)\overline{S_0}}{1 - z\overline{t_0}}$ belong to $\mathcal{H}(S)$.

Moreover, if the conditions (1)–(4) are satisfied then

$$\liminf_{z \rightarrow t_0} \frac{1 - |S(z)|^2}{1 - |z|^2} = \angle \lim_{z \rightarrow t_0} \frac{1 - |S(z)|^2}{1 - |z|^2} = t_0 S_1 \overline{S_0} = \|K_{t_0}\|_{\mathcal{H}(S)}^2 \geq 0$$

and the function K_{t_0} is the boundary reproducing kernel in $\mathcal{H}(S)$ in the sense that

$$\langle f, K_{t_0} \rangle_{\mathcal{H}(S)} = f(t_0) := \angle \lim_{z \rightarrow t_0} f(z). \quad (5.5)$$

Equivalence (2) \Leftrightarrow (3) is the classical Carathéodory-Julia theorem on angular derivatives (see e.g., [Sh1993, Chapter 4]). Note that condition (5.2) is equivalent to the requirement that the function $z \mapsto K_S(z, \zeta)$ stay bounded in the norm of $\mathcal{H}(S)$ as ζ tends to t_0 unrestrictedly in \mathbb{D} .

The equivalences (1) \Leftrightarrow (2) \Leftrightarrow (4) are proved in [S1994, Chapter 6]. Statement (4) is presented in [S1994] in a seemingly weaker form

(4') *There is a number $\lambda \in \mathbb{C}$ such that the function $\frac{1 - S(z)\overline{\lambda}}{1 - z\overline{t_0}}$ belongs to $\mathcal{H}(S)$.*

It then can be shown that there is a unique such λ which turns out to be equal to the boundary limit S_0 as in (5.3).

The list of equivalent conditions in Theorem 5.3 can be extended by several other ones. The extended list of such equivalences will be presented in the context of a more general question: given a Schur-class function S , given a point $t_0 \in \mathbb{T}$ and given an integer $n \geq 0$, find conditions necessary and sufficient for the existence of boundary limits

$$f_j(t_0) := \angle \lim_{z \rightarrow t_0} \frac{f^{(j)}(z)}{j!} \quad \text{for } j = 0, \dots, n \quad (5.6)$$

and for any function $f \in \mathcal{H}(S)$.

Theorem 5.4. *Let $s \in \mathcal{S}$, $t_0 \in \mathbb{T}$ and $n \in \mathbb{N}$. The following are equivalent:*

- (1) *The boundary limits (5.6) exist for every $f \in \mathcal{H}(S)$.*
- (1') *The boundary limit $\angle \lim_{z \rightarrow t_0} f^{(j_n)}(z)$ exists for every $f \in \mathcal{H}(S)$.*
- (2) *S meets the generalized Carathéodory-Julia condition*

$$\liminf_{z \rightarrow t_0} \frac{\partial^{2n}}{\partial z^n \partial \bar{z}^n} \frac{1 - |S(z)|^2}{1 - |z|^2} < \infty. \quad (5.7)$$

Equivalently, the function $\frac{\partial^n}{\partial \zeta^n} K_S(\cdot, \zeta)$ stays bounded in the norm of $\mathcal{H}(S)$ as ζ tends radially to t_0 .

- (3) The boundary limits $S_j := S_j(t_0)$ exist for $j = 0, \dots, 2n+1$ and are such that $|S_0| = 1$ and the matrix

$$\mathbb{P}_n^S(t_0) := \begin{bmatrix} S_1 & \cdots & S_{n+1} \\ \vdots & & \vdots \\ S_{n+1} & \cdots & S_{2n+1} \end{bmatrix} \Psi_n(t_0) \begin{bmatrix} \bar{S}_0 & \cdots & \bar{S}_n \\ & \ddots & \vdots \\ 0 & & \bar{S}_0 \end{bmatrix} \quad (5.8)$$

is Hermitian, where the first factor is a Hankel matrix, the third factor is an upper triangular Toeplitz matrix and where $\Psi_n(t_0)$ is the upper triangular matrix given by

$$\Psi_n(t_0) = [\Psi_{j\ell}]_{j,\ell=0}^n, \quad \Psi_{j\ell} = (-1)^\ell \binom{\ell}{j} t_0^{\ell+j+1}, \quad 0 \leq j \leq \ell \leq n. \quad (5.9)$$

- (4) The boundary limits $S_j := S_j(t_0)$ exist for $j = 0, \dots, n$ and the functions

$$K_{t_0,j}(z) := \frac{z^j}{(1 - z\bar{t}_0)^{j+1}} - S(z) \cdot \sum_{\ell=0}^j \frac{z^{j-\ell} \bar{S}_\ell}{(1 - z\bar{t}_0)^{j+1-\ell}} \quad (5.10)$$

belong to $\mathcal{H}(K_S)$ for $j = 0, \dots, n$. Equivalently, the limits $S_j := S_j(t_0)$ exist for $j = 0, \dots, n$ and the single function $K_{t_0,n}$ belongs to $\mathcal{H}(K_S)$.

- (4') There exist complex numbers $\lambda_0, \dots, \lambda_n$ such that the function

$$\frac{z^j}{(1 - z\bar{t}_0)^{j+1}} - S(z) \cdot \sum_{\ell=0}^n \frac{z^{n-\ell} \bar{\lambda}_\ell}{(1 - z\bar{t}_0)^{n+1-\ell}}$$

belongs to $\mathcal{H}(K_s)$.

- (5) It holds that

$$\sum_k \frac{1 - |a_k|^2}{|t_0 - a_k|^{2n+2}} + \int_0^{2\pi} \frac{d\mu(\theta)}{|t_0 - e^{i\theta}|^{2n+2}} < \infty \quad (5.11)$$

where the numbers a_k come from the Blaschke product of the inner-outer factorization of S :

$$S(z) = \prod_k \frac{\bar{a}_k}{a_k} \cdot \frac{z - a_k}{1 - z\bar{a}_k} \cdot \exp \left\{ - \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) \right\}.$$

- (6) $(A_S^*)^n K_S(\cdot, 0)$ belongs to the range of $(I - \bar{t}_0 A_S^*)^{n+1}$ where $A_S = R_0|_{\mathcal{H}(S)}$ is the model operator for S .
 (7) There exists a finite Blaschke product b such that

$$S(z) = b(z) + o(|z - t_0|^{2n+1}). \quad (5.12)$$

as z tends to t_0 nontangentially.

- (8) *Asymptotic equality (5.12) holds for a rational function b which is unimodular on \mathbb{T} (i.e., b is the ratio of two finite Blaschke products).*

Moreover, if conditions (1)–(8) are satisfied, and hence all, then:

- (a) *The matrix (5.8) is positive semidefinite and equals*

$$\mathbb{P}_n^S(t_0) = [\langle K_{t_0,i}, K_{t_0,j} \rangle_{\mathcal{H}(S)}]_{i,j=0}^n. \quad (5.13)$$

- (b) *The functions (5.10) are boundary reproducing kernels in $\mathcal{H}(S)$ in the sense that*

$$\langle f, K_{t_0,j} \rangle_{\mathcal{H}(S)} = f_j(t_0) := \lim_{z \rightarrow t_0} \frac{f^{(j)}(z)}{j!} \quad \text{for } j = 0, \dots, n. \quad (5.14)$$

Statements (2) \Leftrightarrow (4) \Leftrightarrow (4'), implication (1) \Rightarrow (3) and statements (a) and (b) were proved in [BK2006]; implication (3) \Rightarrow (1) and equivalences (1) \Leftrightarrow (7) \Leftrightarrow (8) appear in [BK2009]. Equivalences (1) \Leftrightarrow (5) \Leftrightarrow (6) were established in [AC1970] for S inner and extended in [FM2008] to general Schur-class functions. The implication (1) \Rightarrow (1') is trivial while the converse implication will be clarified in Lemma 5.5 below. Finally, it is not hard to see that for $n = 0$, the statements (1)–(4') in Theorem 5.4 amount to the respective statements in Theorem 5.3.

5.1. Boundary interpolation. Theorem 5.4 suggests a boundary interpolation problem:

BP $_{\mathcal{H}(S)}$: *Given a Schur-class function S satisfying the Carathéodory-Julia condition (5.7) at $t_0 \in \mathbb{T}$ (or one of the equivalent conditions from Theorem 5.3) and given complex numbers f_0, \dots, f_n , find all $f \in \mathcal{H}(S)$ such that $\|f\|_{\mathcal{H}(S)} \leq 1$ and*

$$f_j(t_0) := \angle \lim_{z \rightarrow t_0} \frac{f^{(j)}(z)}{j!} = f_j \quad \text{for } j = 0, \dots, n. \quad (5.15)$$

According to Theorem 5.4, condition (5.7) guarantees that all the boundary limits in (5.15) exist as well as the boundary limits $S_j := S_j(t_0)$ exist for $j = 0, \dots, 2n + 1$. Introduce the matrices

$$T = \begin{bmatrix} \bar{t}_0 & 1 & \dots & 0 \\ 0 & \bar{t}_0 & & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \bar{t}_0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} E \\ N \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \bar{S}_0 & \bar{S}_1 & \dots & \bar{S}_n \\ \bar{f}_0 & \bar{f}_1 & \dots & \bar{f}_n \end{bmatrix}. \quad (5.16)$$

Simple matrix computations show that

$$\begin{bmatrix} E \\ N \end{bmatrix} (I - zT)^{-1} = \begin{bmatrix} \frac{1}{1 - z\bar{t}_0} & \cdots & \frac{z^n}{(1 - z\bar{t}_0)^{n+1}} \\ \frac{\bar{S}_0}{1 - z\bar{t}_0} & \cdots & \sum_{\ell=0}^n \frac{\bar{S}_\ell z^{n-\ell}}{(1 - z\bar{t}_0)^{n+1-\ell}} \end{bmatrix}.$$

Multiplying the latter equality by $\begin{bmatrix} 1 & -S(z) \end{bmatrix}$ on the left and taking into account formulas (5.10) for $K_{t_0,j}$ one gets

$$F^S(z) := (E - S(z)N)(I - zT)^{-1} = \begin{bmatrix} K_{t_0,0}(z) & K_{t_0,1}(z) & \cdots & K_{t_0,n}(z) \end{bmatrix}. \quad (5.17)$$

The next task is to show that the problem $\mathbf{AIP}_{\mathcal{H}(S)}$ with $\mathcal{X} = \mathbb{C}^{n+1}$ and $\{S, T, E, N, \mathbf{y}\}$ taken in the form (5.16) is equivalent to the problem $\mathbf{BP}_{\mathcal{H}(S)}$. First it will be shown that the data is $\mathbf{AIP}_{\mathcal{H}(S)}$ -admissible.

The first requirement in Definition 4.1 is self-evident since $(I - zT)^{-1}$ is a rational function with no poles inside \mathbb{D} . However, it is worth noting that the pair (E, T) is not output-stable and so $\mathbf{BP}_{\mathcal{H}(S)}$ cannot be embedded into the scheme of the problem $\mathbf{IP}_{\mathcal{H}(S)}$.

For a generic vector $x = \text{Col}_{0 \leq j \leq n} x_j$ in $\mathcal{X} = \mathbb{C}^{n+1}$, It follows from (5.17),

$$F^S(z)x = \sum_{j=0}^n K_{t_0,j}(z)x_j \quad (5.18)$$

and the latter function belongs to $\mathcal{H}(S)$ by statement (4) in Theorem 5.4. It then follows from (5.13) that $\langle \mathbb{P}_n^S(t_0)x, x \rangle_{\mathcal{X}} = \|F^Sx\|_{\mathcal{H}(S)}^2$ and therefore $\mathbb{P}_n^S(t_0) = (F^S)^{[*]}F^S$. It was shown in [BK2006] that the structured matrix $\mathbb{P}_n^S(t_0)$ of the form (5.8) satisfies the Stein identity

$$\mathbb{P}_n^S(t_0) - T^*\mathbb{P}_n^S(t_0)T = E^*E - N^*N.$$

Therefore, the data set $\{S, T, E, N, \mathbf{y}\}$ is $\mathbf{AIP}_{\mathcal{H}(S)}$ -admissible. Observe next that by (5.14) and (5.18), for every $f \in \mathcal{H}(S)$ the following condition holds:

$$\langle M_{F^S}^{[*]}f, x \rangle_{\mathcal{X}} = \langle f, M_{F^S}x \rangle_{\mathcal{H}(S)} = \sum_{j=0}^n f_j(t_0)\bar{x}_j.$$

On the other hand, for \mathbf{y} defined in (5.16), $\langle \mathbf{y}^*, x \rangle_{\mathcal{X}} = \sum_{j=0}^n f_j\bar{x}_j$. Therefore interpolation conditions (5.15) are equivalent to the equality

$$\langle M_{F^S}^{[*]}f, x \rangle_{\mathcal{X}} = \langle \mathbf{y}^*, x \rangle_{\mathcal{X}}$$

holding for every $x \in \mathcal{X}$, i.e., to equality $M_{F^S}^{[*]}f = \mathbf{y}^*$. Thus the problem $\mathbf{AIP}_{\mathcal{H}(S)}$ with the data set $\{S, T, E, N, \mathbf{y}\}$ taken in the form (5.16), is equivalent to the $\mathbf{BP}_{\mathcal{H}(S)}$. In particular, the problem $\mathbf{BP}_{\mathcal{H}(S)}$ has a

solution if and only if $\mathbb{P}_n^S(t_0) \geq \mathbf{y}^* \mathbf{y}$ and all solutions to the problem are parametrized as in Theorem 4.9.

5.2. The vector-valued case. In the vector-valued setting, two additional issues need to be addressed. Firstly, if $\dim \mathcal{Y} = \infty$, one should specify the topology with respect to which the boundary limits should converge. This issue is easily resolved due to the following result; see [BK2008b, Lemma 2.1] for the proof.

Lemma 5.5. *Let $n \in \mathbb{N}$, $t_0 \in \mathbb{T}$, let f be an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ -valued function analytic on $\mathcal{U}_{t_0, \varepsilon} = \{z \in \mathbb{D} : 0 < |z - t_0| < \varepsilon\}$ and assume that for any $\alpha \in (0, \frac{\pi}{2})$ there exists $\gamma_\alpha < \infty$ such that*

$$\|f^{(n)}(z)\| \leq \gamma_\alpha \quad \text{for all } z \in \{z \in \mathcal{U}_{t_0, \varepsilon} : |\arg(z - t_0)| < \alpha\}.$$

Then the uniform limits $\angle \lim_{z \rightarrow t_0} f^{(j)}(z)$ exist for $j = 0, \dots, n-1$.

In particular, the statement holds if the weak limit $\angle \lim_{z \rightarrow t_0} f^{(n)}(z)$ exists.

Thus, once the existence of the nontangential boundary limit (even in the weak sense) for the n -th derivative of any function $f \in \mathcal{H}(S)$ is settled, the existence of strong nontangential boundary limits for derivatives of lower order will be settled automatically.

Secondly, in the vector-valued case, one may want to guarantee the existence of the full or just of a tangential boundary limit. The formulation which incorporates both options appears to be the following.: *Given a Schur-class function $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ and given an $\mathcal{L}(\mathcal{Y}, \mathcal{G})$ -valued polynomial*

$$A(z) = \sum_{j=0}^n A_j(z - t_0)^j, \quad (5.19)$$

find conditions which are necessary and sufficient for the existence of strong boundary limits

$$\angle \lim_{z \rightarrow t_0} (Af)^{(j)}(z) \quad \text{for } j = 0, \dots, n \quad (5.20)$$

and for any function $f \in \mathcal{H}(S)$. The answer is given in the following theorem.

Theorem 5.6. *The strong limits (5.20) exist for any function $f \in \mathcal{H}(S)$ id and only if*

$$\liminf_{z \rightarrow t_0} \left\langle \frac{\partial^{2n}}{\partial z^n \partial \bar{z}^n} \left(A(z) \frac{I_{\mathcal{Y}} - S(z)S(z)^*}{1 - |z|^2} A(z)^* \right) g, g \right\rangle < \infty \quad (5.21)$$

for every $g \in \mathcal{G}$. If this is the case, then the limits

$$b_j := \angle \lim_{z \rightarrow t_0} \frac{(AS)^{(j)}(z)}{j!} \quad (j = 0, \dots, n) \quad (5.22)$$

exist in the strong sense and the limit

$$P = \angle \lim_{z \rightarrow t_0} \frac{\partial^{2n}}{\partial z^n \partial \bar{z}^n} \left(A(z) \frac{I_{\mathcal{Y}} - S(z)S(z)^*}{1 - |z|^2} A(z)^* \right) \quad (5.23)$$

exists in the weak sense.

For the proof and for more equivalent reformulations and consequences of the Carathéodory-Julia condition (5.21) (that is, for the operator-valued version of the Carathéodory-Julia theorem), a good reference is [BK2008b]; see also [BD2006] for the matrix-valued case. Note that the case $n = 0$ in the matrix-valued setting was studied earlier in [DDy1984, Ko1985] and [Dym1980, Section 8],

Observe that upon choosing $A(z) \equiv a_0 \in \mathcal{L}(\mathcal{Y}, \mathcal{G})$ one can derive from Theorem 5.6 the necessary and sufficient condition for the existing of the boundary limit $\angle \lim_{z \rightarrow t_0} a_0 f^{(n)}(z)$ for all $f \in \mathcal{H}(S)$. Finally, here is a formulation of the vector-valued analog of the problem $\mathbf{BP}_{\mathcal{H}(S)}$ from the previous section:

Given a Schur-class function $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ and an $\mathcal{L}(\mathcal{Y}, \mathcal{G})$ -valued polynomial A subject to the Carathéodory-Julia condition (5.21) at $t_0 \in \mathbb{T}$ and given vectors $f_0, \dots, f_n \in \mathcal{Y}$, find all $f \in \mathcal{H}(S)$ such that $\|f\|_{\mathcal{H}(S)} \leq 1$ and

$$f_j(t_0) := \angle \lim_{z \rightarrow t_0} \frac{f^{(j)}(z)}{j!} = f_j \quad \text{for } j = 0, \dots, n. \quad (5.24)$$

It turns out that as in the scalar-valued case, this problem is equivalent to the problem $\mathbf{AIP}_{\mathcal{H}(S)}$ with $\mathcal{X} = \mathcal{G}^{n+1}$ and with the data set

$$T = \begin{bmatrix} \bar{t}_0 I_{\mathcal{G}} & I_{\mathcal{G}} & \dots & 0 \\ 0 & \bar{t}_0 I_{\mathcal{G}} & & \vdots \\ \vdots & \ddots & \ddots & I_{\mathcal{G}} \\ 0 & \dots & 0 & \bar{t}_0 I_{\mathcal{G}} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} E \\ N \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} a_0^* & a_1^* & \dots & a_n^* \\ b_0^* & b_1^* & \dots & b_n^* \\ f_0^* & f_1^* & \dots & f_n^* \end{bmatrix}$$

where the operators $b_0, \dots, b_n \in \mathcal{L}(\mathcal{U}, \mathcal{G})$ are defined in (5.22). This data set turns out to be $\mathbf{AIP}_{\mathcal{H}(S)}$ admissible;. Details are omitted here; note only that the operator $P = (F^S)^{[*]} F^S$ appears be equal to that in (5.24),

6. CONCLUDING REMARKS

The preceding sections give an overview of some of the most recent applications of de Branges-Rovnyak spaces to a variety of problems in function theory, in particular, in interpolation theory. As the following examples illustrate, there is still ongoing work pushing the theory in still more directions.

6.1. Canonical de Branges-Rovnyak functional-model spaces: multivariable settings. Realization of a Schur-class function as the transfer function of a canonical functional-model colligation having additional metric properties (e.g., coisometric, isometric, or unitary), has been extended to settings where the unit disk playing the role of the underlying domain is replaced by a more general domain \mathcal{D} in \mathbb{C}^d ; see [BB2012c] for the case of the unit ball \mathbb{B}^d in \mathbb{C}^d , [BB2012b] for the case of the unit polydisk \mathbb{D}^d , [BB2012a] for the case of a general domain with matrix polynomial defining function.

6.2. Extensions to Kreĭn space settings. Much of the theory of de Branges-Rovnyak spaces actually extends to Pontryagin and Kreĭn-space settings, where Hilbert spaces coming up in various places are allowed to be Kreĭn spaces (i.e., the space is a direct sum of a Hilbert space and an anti-Hilbert space), or at least Pontryagin spaces (where the anti-Hilbert space is finite dimensional).

The AIP approach to interpolation has been extended to the Kreĭn-space setting in work of Derkach [De2001, De2003]; this includes a Pontryagin-space formulation of the Nikolskii-Vasyunin model space $\tilde{\mathcal{D}}(S)$ in terms of Kreĭn-Langer representations.

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